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Saturation Index of π(D(r,s))

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Abstract- Adin and Roichman [1] introduced the concept of permutation graphs and Peter Keevash, Po-Shen Loh and Benny Sudakov [2] identified some permutation graphs with maximum number of edges. Ryuhei Uehara, Gabriel Valiente, discussed on Linear structure of Bipartite Permutation Graphs and the Longest Path Problem [3]. If i, j belongs to a permutation on p symbols {1, 2, …, p} and i is less than j then there is an edge between i and j in the permutation graph if i appears after j in the sequence of permutation. So the line of i crosses the line of j in the permutation. Hence there is a one to one correspondence between crossing of lines in the permutation and the edges of the corresponding permutation graph. In this paper we found the conditions for a permutation to realize a double star and comprehend the algorithm to determine the satuation index of the permutation.AMS Subject Classification (2010): 05C35, 05C69, 20B30.

Key Words- Permutation Graphs, Product of permutations, Double Star, Saturation Numberand Saturation index of a permutation

I. INTRODUCTION

Definition 1.1: Line Representation [4]
Let π be a permutation on p symbolsA ={a1,a2,,ap}where π(ai) = ai',1≤ i ≤ p and |ai+1 – ai| = c, c > 0 , 1≤ i < p. Then the sequence of permutation s(π) is given by {a_1', a_2', …, a_p'}. When the elements are ordered in a line L_1 and the elements of s(π) are ordered in a line L_2 parallel to L_1, then a line joining a_i in L_1 and a_i in L_2 is known as line representation of a_i in π and is denoted by l_i.

Definition 1.2:Residue [4]
Let $a_i,a_j \in A$ Then the residue of a_i and a_j is denoted by $Res(a_i,a_j)$ and is given by $(a_i-a_j)(\pi^{-1}(a_i)-\pi^{-1}(a_j))$.
Note: $Res(a_i,a_j) = Res(a_j,a_i)$.

Definition 1.3:Crossing of lines [4]
Let l_i and l_j represent the lines corresponding to the elements a_i and a_j in π. Then l_i and l_j cross each other if $Res(a_i,a_j) < 0$.

Definition 1.4:Permutation Graph [4,6]
Let π be a permutation given by $\pi(a_i) = a_i$', for all 1≤ i ≤ p. Then the π-permutation graph G_π is given by $G_\pi = (V_\pi, E_\pi)$, where $V_\pi = A$, and $a_ia_j \in E_\pi$ if $Res(a_i, a_j) < 0$.Equivalently if l_i crosses l_j then $a_ia_j \in E_\pi$.
A graph G is a permutation graph if there exists π such that $G_\pi \equiv G$. (i.e) a graph is a permutation graph if it is realisable by a permutation π . Otherwise it is not a permutation graph

The following program identifies the lines that cross each other

```
Program:A C(10)
    Read p
    For (i = 1; i<=p; i++)
    Assign a[i] =i
        Read b[i]
            End for
            For (i = 1; i<=p; i++)
        For (j = i+1; j<=p; j++)
        For (k = 1; k<=p; k++)
        If a[i] =b[k] then
       c[i] =a[k]
         Else
         Next k
         End if
          End for
          For (k = 1; k<=p; k++)
                If a[j] =b[k] then
                   c[j]=a[k]
            Else
            Next k
            End if
End For
 End For
 End For
Program B crossing
Call Program A c(10)
For (i = 1; i<=p; i++)
 For (j = i+1; j<=p; j++)
 If c[i]>c[j] then
            Print 'a[i] and a[j] cross each other
            Else
              Print 'a[i] and a[j] do not cross each other
              End if
        Next j
    Next i
```

Definition 1.5:Neighbourhood, Degree [4]

The neighbourhood of a_i in π is a set of all elements of π whose lines cross l_i and is denoted by $N_\pi(a_i)$ equal to = {$a_r \in \pi$ / $Res(a_i, a_r) < 0$}. The degree of a_i in G_π is denoted by $d_\pi(a_i) = |N_\pi(a_i)|$ which is the number of lines that cross l_i in π.

The following program determines the degree of elements of π in G_π :

```
Program:B Degree of the elements of π d(10)
  Call Program A c(10)
  For (i=1; i<=p; i++)
  For (j=i+1; j<=p; j++)
            If c[i] > c[j] then
  d[i] = d[i] +1
  d[j] = d[j]+1
            Else
        End For
  End For
  For (k=1; k<=p; k++)
      Printf "Degree of a[k] = d[k]"
```

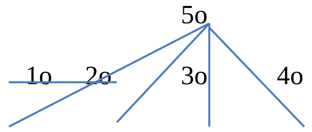
End For

Example 1.1:

Let π be a permutation on {1,2,3,4,5} given by $\pi(1) = 5$; $\pi(2) = 2$; $\pi(3) = 1$; $\pi(4) = 3$ and $\pi(5) = 4$.

Then $G_\pi = G(V_\pi, E_\pi)$ where $V_\pi = \{1,2,3,4,5\}$ and $E_\pi = \{(1,2),(1,5),(2,5),(3,5),(4,5)\}$.

The following graph represents the permutation graph on given π



Here $d_\pi(1) = 2$; $d_\pi(2) = 2$; $d_\pi(3) = 1$; $d_\pi(4) = 1$; $d_\pi(5) = 4$

Definition 1.6: Double Star [7]

Let π be a permutation on a finite set $A = \{a_1, a_2, a_3, \ldots, a_p\}$. Then G_π realises a double star D(r,s), r+s=p, if (i) there exists exactly 2 adjacent elements of degree greater than 1(ii) p-2 elements of degree equal to 1.

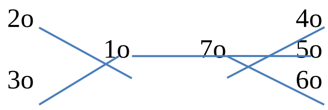
Example 1.2

Let π be a permutation on {1,2,3,4,5,6,7} given by $\pi(1) = 2$; $\pi(2) = 3$; $\pi(3) = 7$; $\pi(4) = 1$; $\pi(5) = 4$; $\pi(6) = 5$ and $\pi(7) = 6$.

Then $G_\pi = G(V_\pi, E_\pi)$ where

$V_\pi = \{1, 2,3,4,5,6,7\}$ and

$E_\pi = \{(1,2),(1,3),(1,7),(4,7),(5,7),(6,7)\}$. The following graph represents D(3,4) which is the permutation graph on given π.



II. PERMUTATION OF A DOUBLE STAR

Definition 2.1: Product of Permutations [8]

Let π be a permutation on p symbols $A = \{a_1, a_2, \ldots, a_p\}$ where $\pi(a_i) = a_i'$, $1 \le i \le p$ and $|a_{i+1} - a_i| = c$, $c > 0$, $1 \le i < p$. Then π^2 be defined as a product of π with itself where $\pi^2(a_i) = \pi(\pi(a_i))$. $\pi^3(a_i) = \pi(\pi^2(a_i))$ for all $1 \le i \le p$. In general, $\pi^m(a_i) = \pi(\pi^{m-1}(a_i))$, $1 \le m \le p$, $1 \le i \le p$. $d_{\pi^m}(a_i) = |N_{\pi^m}(a_i)|$.

Power Condition of a Permutation 3

Let π be a permutation on p symbols $A = \{a_1, a_2, \ldots, a_p\}$ where $\pi(a_i) = a_i'$, $1 \le i \le p$, $\pi(a_r) = a_p$; $\pi(a_{r+1}) = a_1$; $2 \le r \le p-2$; and $|a_{i+1} - a_i| = c$, $c > 0$, $1 \le i < p$. Then π^m, $1 \le m \le p$ is defined as follows:

(i) $\pi^m(a_i) = a_{i+m}$, $1 \le m \le r-1$; $1 \le i \le r-m$

(ii) $\pi^m(a_j) = a_{j-m}$, $1 \le m \le p-r-1$; $r+m < j \le p$

(iii) $\pi^m(a_{r-m+k}) = a_{p+1-k}$,
$1 \le k \le p-r$, $k \le m \le r+k-1$

(iv) $\pi^m(a_{m-k+1}) = a_{p+1-k}$,
$1 \le k \le p-r$; $r+k \le m \le p$

(v) $\pi^m(a_{r-m+k}) = a_{k-p+r}$,
$p-r+1 \le k \le p$; $k \le m \le p$

(vi) $\pi^m(a_{m+r-k}) = a_{k+1}$,
$0 \le k \le r-1$; $k+1 \le m \le p-r+k$

Lemma2.1:

Let π be a permutation on a finite set $A = \{a_1, a_2, a_3, \ldots, a_p\}$ satisfying PCP3. When m=1, π realises D(r,s). Equivalently,if π be a permutation on a finite set $A = \{a_1, a_2, a_3, \ldots, a_p\}$ given by $\pi(a_i) = \pi(a_{i+1})$, $1 \le i \le r-1$; $\pi(a_r) = a_p$; $\pi(a_{r+1}) = a_1$; $\pi(a_j) = a_{j-1}$, $r+2 \le j \le p$,where $2 \le r \le p-2$ and $|a_{k+1} - a_k| = c$, $c > 0$, $0 < k < p-2$, then G_π is a Double Star.

Proof:

Let π be a permutation on a finite set $A = \{a_1, a_2, a_3, \ldots, a_p\}$ given by $\pi(a_i) = \pi(a_{i+1})$, $1 \le i \le r-1$; $\pi(a_r) = a_p$; $\pi(a_{r+1}) = a_1$; $\pi(a_j) = a_{j-1}$, $r+2 \le j \le p$, where $2 \le r \le p-2$ and $|a_{k+1} - a_k| = c$, $c > 0$, $0 < k < p-2$.

(i) We know that $a_1 < a_r$ and $\pi^{-1}(a_1) = a_{r+1}$ and $\pi^{-1}(a_r) = a_{r-1}$, by hypothesis.

Hence Res $(a_1, a_r) < 0$.

(ii) $a_1 < a_i$, $\pi^{-1}(a_i) \le a_{r-2}$ fori $\le$ r-1implies Res$(a_1, a_i) < 0$, for all $i \le r-1$.

Hence $d_\pi(a_1) = r > 1$.

(iii) $a_j < a_p$, $\pi^{-1}(a_j) \le a_{j+1}$ for $r+2 \le j$ implies Res $(a_j, a_p) < 0$, for all $r+2 \le j$. Hence $d_\pi(a_p) = p-r > 1$.

(iv) Let $2 \le i, j \le r$. Then $a_i < a_j$, for $i < j$.

$\pi^{-1}(a_i) = a_{i-1}$, and $\pi^{-1}(a_j) = a_{j-1}$ which implies Res$(a_i, a_j) > 0$.

Let $r+1 \le i, j \le p$. Then $a_i < a_j$, for $i < j$.

$\pi^{-1}(a_i) = a_{i+1}$, and $\pi^{-1}(a_j) = a_{j+1}$ which implies Res$(a_i, a_j) > 0$.We know $a_1 < a_i$, for$1 < i \le r$ and $\pi^{-1}(a_1) = a_{r+1}$and$\pi^{-1}(a_i) = a_{i-1}$ (by hypothesis). Hence $\pi^{-1}(a_1) > \pi^{-1}(a_i)$ which implies Res$(a_1, a_i) < 0$. Similarly $a_i < a_p$, for $r+1 < i \le p$ and

$\pi^{-1}(a_p) = a_r$and $\pi^{-1}(a_i) = a_{i+1}$,byhypothesis. Hence $\pi^{-1}(a_i) > \pi^{-1}(a_p)$ Res $(a_i, a_p) < 0$. Therefore the degree of a_i, $2 \le i \le p-1$ is 1.

(v) We know $a_1 < a_p$.$\pi^{-1}(a_1) = a_{r+1}$ and $\pi^{-1}(a_p) = a_r$,by hypothesis. Hence Res$(a_1, a_p) < 0$. Therefore there exists exactly two elements of π which are adjacent and of degree> 1. Hence (i), (ii), (iii), (iv) and (v) prove that π realises D(r,s) (Definition 2.1).

The following programme identifies the given π as a Double Star or not:

```
Program: D Double Star
 Call Program C
 N=0; M=0
 For ( k =1; k < = p; k ++)
     If d[k]=1 then N=N+1
     Else
     M=M+1
     Next k
  End For
 If (N=p-2 and c [1] > c[p] then
    Printf " π realises a double star "
    Else
    Printf "π does not realise a
 double star" End if
```

III. SATURATION INDEX OF PERMUTATIONS

Definition 3.1: Saturation Numbers [9] A permutation π on p symbols is said to be saturable if $d_{\pi^k}(a_i) = p-1$ for some$1 \le i$

≤ p and 1 ≤ k ≤ p-1. (i.e) if G_{π^k} contains atleast one universal vertex for some 1 ≤ k ≤ p-1.k is known as the saturation number of π, denoted by sn(π).s_π = min { k / k = sn (π)} is called the lower saturation number of π and S_π = max {k / k = sn (π) } is called the upper saturation number of π.

Definition 3.2: Strongly saturable [9]

π is said to be strongly saturable if $d_{\pi^k}(a_i)$ = p-1 for all 1 ≤ i ≤ p and for some 1 ≤ k ≤ p-1.(i.e) if G_{π^k} is a complete graph.

Definition 3.3:Totally saturable [9]

π is said to be totally unsaturable if π is not saturable for any k ≤ p-1 (i.e) G_{π^k} is totally disconnected (or) equivalently, π^k= e and π^t≠ e, for t ≤ k ≤ p-1.π is said to be unilateraly totally unsaturable if π^2 = e.

Definition 3.4:Saturation Index[10]

The saturation index of a permutation is given by ζ (π) = |S_π–s_π |.

Remark: $S_\pi + s_\pi$= p.

Lemma 3.1:

Let π be a permutation on p symbols realising D (r,s), r ≤ s . Then lower saturation number of π is r and the upper saturation number of π is s.

Proof: Let π be a permutation on p symbols realising D(r,s). Hence π(a_r) = a_p; π^2(a_{r-1})=a_p; π^3(a_{r-2}) = a_p; …; π^m (a_{r-m+1}) = a_p; and π(a_{r+1}) = a_1; π^2(a_{r+2})=a_1; π^3(a_{r+3}) = a_1;…; π^m (a_{r+m}) = a_1, by Definition 2.1 and Program C. The saturation number is the value of m when π^m (a_1) = a_p or π^m (a_p) = a_1. (i.e) when r-(m-1) = 1 or r+m = p which impliesr = m or r = p - m = p - r = s. Lower saturation number = min{m , p-m}and upper saturation number = max{m , p-m}. Hence the lower saturation number of π of D (r,s), is m = r and the upper saturation number is p – m = s.

Hence the Lemma.

Theorem 3.1:

Let π be a permutation on p symbols satisfying PCP3. Then the saturation index ζ(π)= p-2r.

Proof:

Let π be a permutation on p symbols satisfying PCP3. Then by Lemma 2.1 π realises a D(r,s), r+s = p and by Lemma 3.1, the upper and lower saturation numbers are r and p-r.

Hence ζ(π)= p-r –r = p-2r.

Hence the theorem.

The saturation numbers of π of D (r,s) are found by the following program:

Program: E Saturation Numbers ofπ(D (r,s))

```
Read p
For (i = 1; i<=p; i++)
Assign a[i] = i
Next i
For (r=2; r<=p-2; r++)
      For ( i=1; i<=p; i++)
b[i] =a[i]
       Next i
For (m=1; m<=p; m++)
u=b [1]
v=b[p]
if (u=a[p]) then
x=m-1
y=p-x
Print "the saturation numbers of π(D(r,s)) are x and y
 Goto end
     Else
        For (i=1; i<=r-1; i++)
b[i] =b [i+1]
        Next i
       For (i=p; i>r+2; i--)
b[r] = v
b[r+1] = u
        Next i
 End For
 End For
  End
```

Example: 3.1

Let p=5; r=2; then $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 1 & 3 & 4 \end{pmatrix}$ and G_π = D (2, 3).

Here π^2(1) = 5 and π^3 (5) = 1.

ζ (π(D(2,3))) =|S_π -s_π|= 3-2 =1.

Example 3.2:

Let p = 8 , r = 3.

Then $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 3 & 8 & 1 & 4 & 5 & 6 & 7 \end{pmatrix}$ and G_π= D (3, 5).

Here π^3(1) = 8 and π^5(8) = 1.

ζ(π(D (3, 5))) = |S_π - s_π| = 5-3 = 2.

Note:

1. Let π be a permutation on {a_1, a_2,..., a_p } such that |a_{i+1} -a_i| = c, c > 0 satisfying PCP1. Then π is strongly saturable and the saturation index of π is zero for even number of elements. For odd p the saturation index is |S_π -s_π| = 1.[9]

2.C_3, C_4, D (2, 2) and $K_{r,s}$ where r = s are unilateraly totally unsaturable. [9]

3. Let π be a permutation on A={ a_1, a_2, …, a_p} such that |a_{i+1} - a_i| = c, c > 0 satisfying PCP2. Then for p ≥ 3, r ≥ 3 π is saturable when r m ≡ 1 mod p or r m ≡ (p-1) mod p. The saturation index of π($K_{1,p-1}$) is p-2. [10]

2. π is not saturable for p such that (p,r) = d ≠1 and is totally unsaturable whenm =p/d.[10]

4. The saturation index of π (D(r, r)) is zero, π (D(r, r+1)) is 1.

6. There exists a permutation π such that G_π ≡ D(r, s) which is a bipartite graph.

7. Open question: Shall we adopt a similar process to establish a PCP to determine the saturation index for general permutation graphs?

IV. CONCLUSION

The permutation graphs in terms of crossing of lines and the sequence of permutations were defined and results on permutations realising bipartite, tripartite, complete and path were discussed by us earlier based on the definitions and results found in [1,2,3,6]. Then the power conditions on permutations for realising a path and complete bipartite graph were established. The saturation indices for the permutations satisfying PCP1 and PCP2 were determined [9,10]. In this paper an algorithm for finding the saturation index of one

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specific class of bipartite permutation graphs, which is double star D(r,s), is discussed by instituting PCP3. An open question is posed at the end of the paper.

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