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Solving Fuzzy Differential Equationsin Runge-Kutta Method of Order Three

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Abstract- In this paper, we study numerical method for Fuzzy differential equations by Runge-Kutta method of order three. The elementary properties of this method are given. We use the extended Runge-Kutta method of order three in order to enhance the order of accuracy of the solution. Thus we can obtain the strong Fuzzy solution.

Keywords- Fuzzy differential equations, Runge-Kutta method of order three, Trapezoidal Fuzzy number.

I. INTRODUCTION

In this paper, we have introduced and studied a new technique forgetting the solution of fuzzy initial value problem. The organized paper is asfollows: In the first three sections, we recall some concepts in fuzzy initial value problem. In sections four and five,we present Runge-Kutta method of order three and its iterative solution forsolving Fuzzy differential equations. The proposed algorithm is illustrated by anexample in the last section.

II. PRELIMINARY

A trapezoidal fuzzy number u is defined by four real numbers $0 < k < \ell < m < n$ where the base of the trapezoidal is the interval [k, n] and itsvertices at $x = \ell$, $x = m$. Trapezoidal fuzzy number will be written as $u = (k, \ell, m, n)$. The membership function for the trapezoidal fuzzy number $u = (k, \ell, m, n)$ is defined as follows :

$$u(x) = \begin{cases} \dfrac{x-k}{\ell - k}, & k \le x \le \ell \\ 1, & \ell \le x \le m \\ \dfrac{x-n}{m-n}, & m \le x \le n \end{cases}$$

The results may be:

$(1)\, u > 0 \quad if \quad k > 0;$

$(2)\, u > 0 \quad if \quad \ell > 0;$

$(3)\, u > 0 \quad if \quad m > 0;$

and $(4)\, u > 0 \quad if \quad n > 0;$

Let us denote $R_F$bythe class of all fuzzy subsets of R (i.e. $u : R \to [0,1]$) satisfying the following properties:

(i) $\forall u \in R_F,$ u is normal, i.e. $\exists x_0 \in R$ with $u(x_0) = 1$

(ii) $\forall u \in R_F$, u is convex fuzzy set(i. e. $u(tx + (1-t)y) \ge \min\{u(x), u(y)\}, \forall t \in [0,1], x, y \in R$)

(iii) $\forall u \in R_F,$ u is upper semi continuous on R; (iv) $\overline{\{x \in R; u(x) > 0\}}$ is compact, where $\bar{A}$ denotes the closure of A. Then R_F is called the space of fuzzy numbers.

Obviously $R \subset R_F$. Here $R \subset R_F$ is understood as $R = \{\chi_{\{x\}}; x \text{ is usual real number}\}$

We define the r-level set, $x \in R$;

$[u]_r = \{x \setminus u(x) \ge r\}, \qquad 0 \le r \le 1;$

clearly $[u]_0 = \{x \setminus u(x) > 0\}$ is compact,

Theorem 2.1
Let $F(t,u,v)$ and $G(t,u,v)$ belong to $C^1(R_F)$ and the partial derivatives of F and G be bounded over R_F. Then for arbitrarily fixed $r, 0 \le r \le 1$, the numerical solutions of $\underline{y}(t_{n+1}; r)$ and $\overline{y}(t_{n+1}; r)$ converge to the exact solutions $\underline{Y}(t;r)$ and $\overline{Y}(t;r)$ uniformly in t.

Theorem 2.2
Let $F(t,u,v)$ and $G(t,u,v)$ belong to $C^1(R_F)$ and the partial derivatives of F and G be bounded over R_F and $2Lh < 1$. Then for arbitrarily fixed $0 \le r \le 1$, the iterative numerical solutions of $\underline{y}^{(j)}(t_n; r)$ (1) and $\overline{y}^{(j)}(t_n; r)$ converge to the numerical solutions $\underline{y}(t_n; r)$ and $\overline{y}(t_n; r)$ in $t_0 \le t_n \le t_N$, when $j \to \infty$.

III. FUZZY INITIAL VALUE PROBLEM

Consider a first-order fuzzy initial value differential equation is given by

$$\begin{cases} y'(t) = f(t, y(t)), & t \in [t_0, T] \\ y(t_0) = y_0 \end{cases} \qquad (3)$$

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We denote the fuzzy function y by $y = [\underline{y}, \overline{y}]$. It means that the r-level set of y(t) for $t \in [t_0, T]$ is

$$[y(t)]_r = [\underline{y}(t;r), \overline{y}(t;r)],$$

$$[y(t_n)]_r = [\underline{y}(t_0;r), \overline{y}(t_0;r)],\ r \in (0,1]$$

we write $f(t,y) = [\underline{f}(t,y), \overline{f}(t,y)]$

Because of $y' = f(t,y)$ we have

$$\underline{f}(t,(y(t);r) = F[t, \underline{y}(t;r), \overline{y}(t;r)]$$

$$\overline{f}(t,(y(t);r) = G[t, \underline{y}(t;r), \overline{y}(t;r)]$$

By using the extension principle, we have the membership function

$$f(t, y(t))(s) = sup\{y(t)(\tau) \setminus s = f(t,\tau)\},\ s \in R$$

so fuzzy number $f(t, y(t))$. From this it follows that

$$[f(t, y(t))]_r = [\underline{f}(t, y(t); r), \overline{f}(t, y(t); r)],\ r \in (0,1]$$

where

$$\underline{f}(t, y(t); r) = min$$

$$\{f(t,u) \mid u \in [y(t)]_r\}$$

$$\overline{f}(t, y(t); r) = max$$

$$\{f(t,u) \mid u \in [y(t)]_r\}$$

Definition - A function $R \rightarrow R_F$ is said to be fuzzy continuous function, if for an arbitrary fixed $t_0 \in R$ and $\in > 0, \delta > 0$ such that

$$|t - t_0| < \delta \Rightarrow D[f(t), f(t_0)] < \in$$

Throughout this paper it is considered that fuzzy functions are continuous in metric *D*. Then the continuity of *f(t,y(t);r)*guarantees the existence of the definition *f(t, y(t); r)* for $t \in [t_0,T]$ and $r \in [0,1]$. Therefore, the functions *G* and *F* can be defined too.

IV. RUNGE-KUTTA METHOD OF ORDER THREE

Consider the initial value problem

$$\begin{cases} y'(t) = f(t, y(t)), & t \in [t_0, T] \\ y(t_0) = y_0 \end{cases}$$

Assuming the following Runge-Kutta method with three slopes

$$y(t_{n+1}) = y(t_n) + W_1K_1 + W_2K_2 + W_3K_3$$

where

$$K_1 = hf(t_n, y(t_n))$$

$$K_2 = hf(t_n + c_2h, y(t_n) + a_{21}K_1)$$

$$K_3 = hf(t_n + c_3h, y(t_n) + a_{31}K_1 + a_{32}K_2)$$

and the parameters $W_1, W_2, W_3, c_2, c_3, a_{21}, a_{31}$ & a_{32} are chosen to make y_{n+1} closer to $y(t_{n+1})$. There are eight parameters to be determined. Now, Taylor's series expansion about t_n gives

$$y(t_{n+1}) = y(t_n) + \frac{h}{1!}y'(t_n) + \frac{h^2}{2!}y''(t_n) + \frac{h^3}{3!}y'''(t_n) + \ldots = \quad (4)$$

$$y(t_n) + \frac{h}{1!}f(t_n, y(t_n)) + \frac{h^2}{2!}[f_t + ff_y]_{tn} + \ldots.$$

$$K_1 = hf_n$$

$$K_2 h\{f_n + \frac{h}{1!}[c_2f_t + a_{21}ff_y]_{tn} + \frac{h^2}{2!}[c_2^2f_{tt} + 2c_2a_{21}ff_{ty} \quad (6)$$

$$+ a_{21}^2 f^2 f_{yy}]_{tn} + \ldots..\} \quad (7)$$

$$K_3 = h$$

$$\{f_n + \frac{h}{1!}[c_3f_t + (a_{31} + a_{32})f_nf_y]_{t_n} + \quad (8)$$

$$\frac{h^2}{2!}\begin{bmatrix} 2(c_2f_t + a_{21}ff_y)a_{32}f_y + c_3^2f_{tt} \\ +2c_3a_{31}f_n + 2c_3a_{32}f_{ty}f_n \\ +(a_{31}^2 + a_{32}^2 + 2a_{31}a_{32})f_n^2f_{yy} \end{bmatrix}_{t_n}$$

+h^3/3! [(3(⟦c_2⟧ ^2 f_tt + 2c_2 a_21 ⟦ff⟧ _ty + ⟦2_21⟧ ^2 f^2 f_yy) a_32 f_y @+ (6c_3 a_32 f_ty + 6a_31 f_n a_32 f_yy)(c_2 f_t + a_21 ⟦ff⟧ _y))]_(t_n)

+… }

Substituting the values of $K_1, K_2 \& K_3$,we get

$$y(t_{n+1}) = y(t_n) + [W_1 + W_2 + W_3]hf_n + h^2[W_2(c_2f_t + a_{21}ff_y) + W_3(c_3f_t + (a_{31} + a_{32})f_nf_y)]_{t_n}$$

+h^3/2 [(W_2 (⟦c_2⟧ ^2 f_tt + 2c_2 a_21 ⟦ff⟧ _ty + ⟦a_21⟧ ^2 f^2 f_yy) +@W_3 ((2(c_2 f_t + a_21 ff_y) a_32 f_y + ⟦c_3⟧ ^2 f_tt + 2c_3 a_31 f_n + @2c_3 a_32 f_n f_ty + (⟦a_31⟧ ^2 + ⟦a_32⟧ ^2 + 2a_31 a_32) ⟦f_n⟧ ^2 f_yy)))]_(⟦t⟧ _n)

+…

(13)

Comparing the coefficients of h, h^2& h^3, we obtain (10)

$$a_{21} = c_2, \qquad a_{31} + a_{32} = c_3, \qquad W_1 + W_2 + W_3 = 1,$$

$$c_2W_2 + c_3W_3 = \frac{1}{2}, \quad c_2^2W_2 + c_3^2W_3 = \frac{1}{3}, \quad c_2a_{32}W_3 = \frac{1}{6} \quad (11)$$

(14)

Then we immediately obtain from the fourth and fifth equations , that $c_2 = \frac{2}{3}$. Similarly the values of the remaining parameters are obtained.

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When $c_2 = c_3$, we get $c_2 = \frac{2}{3}$ and $a_{21} = \frac{2}{3}$. We get the values of the other parameters as $a_{31} = 0, a_{32} = \frac{2}{3}, W_1 = \frac{2}{8}, W_2 = \frac{3}{8}$ & $W_3 = \frac{3}{8}$.

Runge-Kutta method is obtained as

$$y(t_{n+1}) = y(t_n) + \frac{1}{8}[2K_1 + 3K_2 + 3K_3]$$

Where

$$K_1 = hf\left(t_n, y(t_n)\right)$$

$$K_2 = hf\left(t_n + \frac{2h}{3}, y(t_n) + \frac{2}{3}K_1\right)$$

$$K_3 = hf\left(t_n + \frac{2h}{3}, y(t_n) + \frac{2}{3}K_2\right)$$

V. RUNGE-KUTTA METHOD OF ORDER THREE FOR SOLVING FUZZY DIFFERENTIAL EQUATIONS

Let $Y = [\underline{Y}, \overline{Y}]$ be the exact solution and $y = [\underline{y}, \overline{y}]$ be the approximated solution of the fuzzy initial value problem .

Let

$[Y(t)]_r = [\underline{Y}(t;r), \overline{Y}(t;r)], [y(t)]_r = [\underline{y}(t;r), \overline{y}(t;r)]$ Throughout this argument, the value of r is fixed. Then the exact and approximated solution at t_n are respectively denoted by

$$[Y(t_n)]_r = [\underline{Y}(t_n;r), \overline{Y}(t_n;r)],$$
$$[y(t_n)]_r = [\underline{y}(t_n;r), \overline{y}(t_n;r)] (0 \le n \le N).$$

The grid points at which the solution is calculated are

$$h = \frac{T - t_0}{N}, t_i = t_0 + ih, 0 \le i \le N .$$

Then we obtain,

$$\underline{Y}(t_{n+1};r) = \underline{Y}(t_n;r) + \frac{1}{8}[2K_1 + 3K_2 + 3K_3],$$

where $K_1 = hF[t_n, \underline{Y}(t_n;r), \overline{Y}(t_n;r)]$

$$K_2 = hF[t_n + \frac{2h}{3}, \underline{Y}(t_n;r) + \frac{2}{3}K_1, \overline{Y}(t_n;r) + \frac{2}{3}K_1]$$

$$K_3 = hF[t_n + \frac{2h}{3}, \underline{Y}(t_n;r) + \frac{2}{3}K_2, \overline{Y}(t_n;r) + \frac{2}{3}K_2]$$ and

$$\overline{Y}(t_{n+1};r) = \overline{Y}(t_n;r) + \frac{1}{8}[2K_1 + 3K_2 + 3K_3],$$ where

$$K_1 = hG[t_n, \underline{Y}(t_n;r), \overline{Y}(t_n;r)]$$

$$K_2 = hG[t_n, + \frac{2h}{3}, \underline{Y}(t_n;r) + \frac{2}{3}K_1, \overline{Y}(t_n;r) + \frac{2}{3}K_1]$$

(17)

$$K_3 = hG[t_n, + \frac{2h}{3}, \underline{Y}(t_n;r) + \frac{2}{3}K_2, \overline{Y}(t_n;r) + \frac{2}{3}K_2]$$

Also we have

$$\underline{y}(t_{n+1};r) = \underline{y}(t_n;r) + \frac{1}{8}[2K_1 + 3K_2 + 3K_3]$$

where

$$K_1 = hF[t_n, \underline{y}(t_n;r), \overline{y}(t_n;r)]$$

$$K_2 = hF[t_n, + \frac{2h}{3}, \underline{y}(t_n;r) + \frac{2}{3}K_1, \overline{y}(t_n;r) + \frac{2}{3}K_1] \quad (18)$$

(15)

$$K_3 = hF[t_n, + \frac{2h}{3}, \underline{y}(t_n;r) + \frac{2}{3}K_2, \overline{y}(t_n;r) + \frac{2}{3}K_2]$$

and

$$\overline{y}(t_{n+1};r) = \overline{y}(t_n;r) + \frac{1}{8}[2K_1 + 3K_2 + 3K_3],$$ where

$$K_1 = hG[t_n, \underline{y}(t_n;r), \overline{y}(t_n;r)]$$

$$K_2 = hG[t_n, + \frac{2h}{3}, \underline{y}(t_n;r) + \frac{2}{3}K_1, \overline{y}(t_n;r) + \frac{2}{3}K_1]$$

$$K_3 = hG[t_n, + \frac{2h}{3}, \underline{y}(t_n;r) + \frac{2}{3}K_2, \overline{y}(t_n;r) + \frac{2}{3}K_2]$$

Clearly, $\underline{y}(t;r)$ and $\overline{y}(t;r)$ converge to $\underline{Y}(t;r)$ and $\overline{Y}(t;r)$ respectivelywhen ever $h \to 0$

VI. NUMERICAL RESULTS

In this section, the exact solution and approximated solution are obtained byEuler's method and Runge-Kutta method of order three.

Example
Consider the initial value problem

$$\begin{cases} y'(t) = f(t), & t \in [0,1] \\ y(0) = (0.75+0.25r, 1.125 - 0.125r) \end{cases}$$

The exact solution at t=1 is given by

$$Y(1;r) = [(0.75+0.125r)e, (1.125-0.125r)e],$$

$0 \le r \le 1$
Using iterative solution of Runge-Kutta method of order three, we have

$$\underline{y}(0;r) = 0.75 + 0.25r,$$
$$\overline{y}(0;r) = 1.125 - 0.125r \quad (16)$$

And by

$$\underline{y}^{(0)}(t_{i+1};r) = \underline{y}(t_i;r) + h\underline{y}(t_i;r)$$
$$\overline{y}^{(0)}(t_{i+1};r) = \overline{y}(t_i;r) + h\overline{y}(t_i;r)$$

Where i=0,1,2,...N-1 and $h = \frac{1}{N}$. Now, using these equations as an initial guess for following iterative solutions respectively,

$$\underline{y}^j(t_{i+1};r) = \underline{y}(t_i;r) + \frac{1}{8}[2K_1 + 3K_2 + 3K_3],$$

$$K_1 = h\underline{y}(t_i;r)$$

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$$K_2 = h\left(\underline{y}(t_i;r) + \frac{2}{3}K_1\right)$$

$$K_3 = h\left(\underline{y}(t_i;r) + \frac{2}{3}K_2\right)$$

And

$$\overline{y}^j(t_{i+1};r) = \overline{y}(t_i;r) + \frac{1}{8}[2K_1 + 3K_2 + 3K_3],$$

$$K_1 = h\overline{y}(t_i;r)$$

$$K_2 = h\left(\overline{y}(t_i;r) + \frac{2}{3}K_1\right)$$

$$K_3 = h\left(\overline{y}(t_i;r) + \frac{2}{3}K_2\right)$$

And j=1,2,3. Thus, we have $\underline{y}(t_i;r) = \underline{y}^{(3)}(t_i;r)$ and

$$\overline{y}(t_i;r) = \overline{y}^{(3)}(t_i;r), for\ i = 1 \dots N$$

Therefore, $\underline{Y}(1;r) \approx \underline{y}^{(3)}(1;r)$ and $\overline{Y}(1;r) \approx \overline{y}^{(3)}(1;r)$ are obtained.

By minimizing the step size h, the solution by exact method and Runge- Kutta method almost coincides.

Table 1: Exact solution

r	$\underline{Y}$	$\overline{Y}$
0	2.038711371	3.058067057
0.1	2.106668417	3.024088534
0.2	2.174625463	2.990110011
0.3	2.242582508	2.956131488
0.4	2.310539554	2.922152966
0.5	2.378496600	2.888174443
0.6	2.446453646	2.85419592
0.7	2.514410691	2.820217397
0.8	2.582367737	2.786238874
0.9	2.650324783	2.752260351
1	2.718281828	2.718281828

Table 2: Approximated solution

r	$\underline{y}$	$\overline{y}$
0	2.038633	3.057949
0.1	2.106587	3.023972
0.2	2.174542	2.989995
0.3	2.242496	2.956018
0.	2.310451	2.922041
0.5	2.378405	2.888063
0.6	2.446360	2.854086
0.7	2.514314	2.820109
0.8	2.582260	2.786132
0.9	2.650223	2.752154
1	2.718177	2.718177

VII. CONCLUSION

In this paper, numerical method for solving Fuzzy differential equations is considered. A scheme based on thirdorder Runge – Kuttamethod to approximate the solution of fuzzy initial value problem has been formulated. Numerical example shows that the exact and approximate solutions converge when h $\rightarrow$ 0 .

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