

Conditional Matching Preclusion Number of Certain Graphs

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Abstract - The matching preclusion number of a graph is the minimum number of neither edges whose deletion in a graph has a neither perfect matching nor an almost perfect matching. For many interconnection networks, the optimal sets are precisely those induced by a single vertex. Recently the conditional matching preclusion number of a graph was introduced to look for sets beyond those induced by a single vertex. It is defined to be the minimum number of edges whose deletion results in a graph with no isolated vertices and has neither a perfect matching nor almost perfect matching. In this paper we find the conditional matching preclusion number for triangular ladder, C_n with parallel chords, Trampoline Graph, diamond Snake Graph and K- Polygonal Snake Graph.

Keywords: Conditional Matching Preclusion Number, Triangular Ladder, C_n with Parallel Chords, Trampoline Graph, Diamond Snake Graph and K- Polygonal Snake Graph.

I. INTRODUCTION

In graph theory and network analysis, the Robustness of a graph or class of graphs measures its resilience to the removal of edges or vertices. Robustness can be formalized in a variety of ways, depending on how the edges or vertices to be removed are chosen. Robustness is of practical concern to those working with physical networks, including those involved in gene regulation, disease transmission, and the internet. A gene or computer network that is robust with respect to random removal retains much of its functionality in the face of random failures. If the elimination of vertices or edges is directed in some fashion, then robustness implies resilience to targeted attacks. This kind of robustness is also of interest to epidemiologists who want to break up disease transmission networks by targeted vaccinations. In certain applications, every vertex requires a special partner at any given time and the matching preclusion number measures the robustness of this requirement in the event of link failures.

A matching in a graph $G = (V, E)$ is a subset M of edges, no two of which have a vertex in common. The vertices belonging to the edges of a matching are said to be saturated by the matching. The others are unsaturated. A perfect matching in a graph is a set of edges such that every vertex is incident with exactly one edge in this set. An almost perfect matching in a graph is a set of edges such that every vertex except one is

incident with exactly one edge in this set, and the exceptional vertex is incident to none. So if a graph has a perfect matching, then it has an even number of vertices; if a graph has almost-perfect matching, then it has an odd number of vertices. The matching preclusion number of a graph G , denoted by $mp(G)$, is minimum number of edges whose deletion leaves the resulting graph without a perfect matching or almost perfect matching. Any such optimal set is called an optimal matching preclusion set. $mp(G) = 0$, if G has neither a perfect matching nor an almost-perfect matching.

The concept of matching preclusion was introduced by Brigham et al. [1] and further studied by Cheng and Liptak [3,7]. In [1], the matching preclusion number was determined for three classes of graphs namely complete bipartite graphs and hypercube. Hence in these interconnection networks, it is desirable to have the property that the only optimal matching preclusion sets are those whose elements are incident to a single vertex. The conditional matching preclusion number of a graph G , denoted by $mp_1(G)$, with n vertices is the minimum number of edges whose deletion results in a graph without an isolated vertex and does not have a perfect matching if n is even or almost-perfect matching if n is odd. Any such optimal set is called an optimal conditional matching preclusion set. $mp_1(G) = 0$ if G has neither a perfect matching nor an almost-perfect matching. If a conditional matching preclusion set does not exist then also $mp_1(G) = 0$. The conditional matching preclusion for hypercube-like interconnection networks has been studied by Park et al. [8]. Further Cheng et al. determine the conditional matching preclusion number for the arrangement graphs [5], tori and related Cartesian products [6], certain bipartite graphs [10] and twisted cubes [11].

In chemistry conditional matching preclusion is termed as anti-kekule number. In organic molecule graphs, perfect matching's corresponding to kekule structures, playing an important role in analysis of the resonance energy and stability of hydrocarbon compounds. Cyvin and Gutman systematically gave [12] detailed enumeration formulas for kekule structures of various types of benzenoids. Kardos et al. showed [13] that fullerene graphs have exponentially many kekule structures. Anti-kekule structures. An anti-kekule set of G with kekule structures is the set S such that $G-S$ is a connected graph and it has no kekule structures. An anti-kekule set of the smallest

cardinality is called a minimal anti-kekule set, and its cardinality is the anti-kekule number of G and it is denoted by $ak(G)$. The anti-kekule number has been studied in [12-18].

II. PRELIMINARIES

Lemma 2.1. [4] Let G be a graph with an even number of vertices. Then $mp(G) \leq \delta(G) \leq$ where $\delta(G)$ is the minimum degree of G .

In lemma 2.1 without the condition of no isolated vertices an isolated vertex will be the basic obstruction and so deleting all edges incident to G will produce a trivial matching preclusion set., Now for a graph with no isolated vertex a basic obstruction to a perfect matching will be the existence of a path $u-v-w$ where the degree of u and the degree of w are 1. So to produce such an obstruction set, one can pick any path $u-v-w$ the original graph and delete all edges incident to either u or w but not v .

Lemma 2.2.[4] Let G be a graph with an even number of vertices. Suppose $\delta \geq 3$, then $mp_1(G) \leq V_e(G)$ where $V_e(G) = \min\{d_G(u) + d_G(v) - 2 - y_G(u,v) : u \text{ and } v \text{ are distance } 2 \text{ apart}\}$ and $d_G(\cdot)$ is the degree function $y_G(u,v) = 1$ if u and v are adjacent and 0 otherwise.

Now we define an upper bound for conditional matching preclusion number of a general graph G with minimum degree $\delta \geq 2$. In the following lemma we deal with a larger family of graphs with $\delta \geq 2$

Lemma 2.3: Let G be a graph $\delta \geq 2$ and let $u-v-w$ is a path of length 3, such that $d_G(v) > d_G(u)$ and $d_G(v) \geq d_G(w)$. Then $mp_1(G) \leq V_e(G)$. We note that the condition $\delta \geq 2$ and let $u-v-w$ is a path of length 3, such that $d_G(v) > d_G(u)$ and $d_G(v) \geq d_G(w)$ is to ensure that the resulting graph (after edges were removed) has no isolated vertices. If $mp_1(G) = V_e(G)$ then G is called conditionally maximally matched. We call a solution of the form induced by V_e a conditional matching preclusion set.

III. MAIN RESULTS

3.1. Triangular Ladder

Definition: A triangular ladder C_n $n \geq 2$ is a graph obtained by completing the ladder $L_n \approx P_n \times P_2$ by edges $u_i v_{i+1}$ for $1 \leq n \leq n - 1$.(see Figure 1) [19].

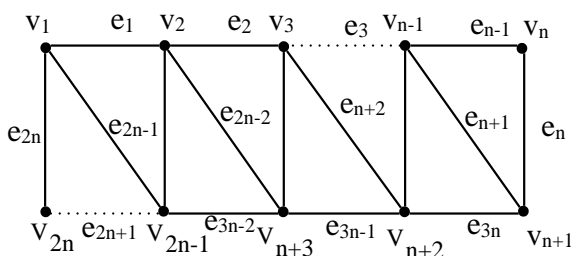


Figure 1: Triangular ladder graph.

Theorem 1: Let G be the Triangular ladder graph. Then $mp_1(G) = 3$.

Proof: Let M_V be the set of all vertical edges and M_H be the set of all horizontal edges and M_C be the set of all cross edges in triangular ladder. Thus each of V, H, C contains a perfect matching's $M_V, M_H,$ and M_C respectively. We now claim that $mp_1(G) > 2$. Suppose e_1 and e_2 edges are in triangular ladder. Clearly e_1 and e_2 belong to almost two categories of edges. But there exists a perfect matching in $G \setminus \{e_1, e_2\}$ the third category of edges. This implies $mp_1(G) > 2$. Hence $mp_1(G) \geq 3$.

We have only to prove that $mp_1(G) \leq 3$. Now $u-v-w$ is a path in triangular ladder with $d_G(u) = 2, d_G(v) = 3$ and $d_G(w) = 4$. Remove the edges one end incident to u but not at v , similarly remove the edges one end incident to w but not at v . The resulting graph has uv and vw pendant edges. If the pendant uv is chosen in a perfect matching, the w vertex is left unsaturated and vice versa. Therefore $mp_1(G) = 3$.

IV. CYCLE C_n WITH PARALLEL CHORDS

Definition: A graph is said to be the cycle C_n with parallel chords if it is obtained from cycle $C_n = \{v_0, v_1, \dots, v_{n-1}\}$ by adding the parallel chords,

$v_{n-1}, v_2 v_{n-2}, v_3 v_{n-3}, \dots, v_{\lfloor n/2 \rfloor - 1} v_{\lfloor n/2 \rfloor + 1}$ according as n is odd or even.(see Figure 2)[19].

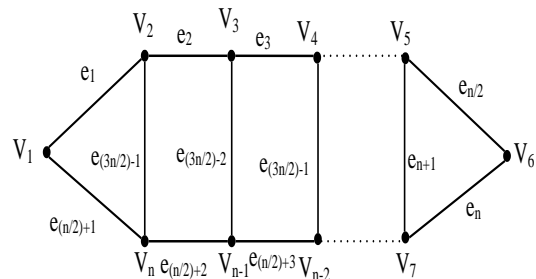


Figure 2: C_n with parallel chords.

Theorem 2: Let G be the C_n with parallel chords. Then $mp_1(G) = 2$.

Proof: We now claim that $mp_1(G) > 1$. Let H_i be the Hamiltonian cycle of C_n with parallel chords. Clearly the cycle H_i is of even length. Let M_1^i consist of alternate edges in H_i and let M_2^i consist of remaining edges in H_i , such that $M_1 = \cup_{i=1}^n M_1^i$ and $M_2 = \cup_{i=1}^n M_2^i, 1 \leq i \leq n$. Clearly M_1 and M_2 are perfect matching's. Let $e \in H_i$ then either $e \in M_1^i$ or $e \in M_2^i$. Without loss generality $e \in M_1^i$ then M_2^i is a perfect matching. Let E' be the set of edges in $E(G \setminus M_1 \cup M_2)$. Suppose $e \in E'$, then M_1 and M_2 not containing e . This implies $mp_1(G) > 1$. Hence $mp_1(G) \geq 2$

We have only to prove that $mp_1(G) \leq 2$. Now $u-v-w$ is a path in C_n with parallel chords with $d_G(u) = 2, d_G(v) = 3$ and $d_G(w) = 3$.

Remove the edges one end incident to u but not at v , similarly remove the edges one end incident to w but not at v . The resulting graph has uv and vw pendant edges. If the pendant uv is chosen in a perfect matching, the w vertex is left unsaturated and vice versa. Therefore $mp_1(G) = 2$.

V. TRAMPOLINE GRAPH

Definition: A chordal graph is a graph which every cycle of length ≥ 4 has a chord. A sun or a trampoline is a chordal graph which has a Hamiltonian cycle $x_1y_1x_2y_2 \dots \dots x_ny_nx_1$ where each y_i $1 \leq i \leq n$ is a degree 2. A complete sun or complete trampoline is a graph in which $\{x_1, x_2, \dots \dots x_n\}$ is a clique we call this sun as a 0-sun. A 1-sun is obtained from 0-sun by including triangles on the edges $(x_1y_1)(y_1x_2)(x_2y_2) \dots \dots y_r x_r$ if this procedure is repeated t times then we obtain t -layered sun or simply a t -sun. (See Figure 3)[21].

Theorem 3: Let G be the Trampoline Graph. Then $mp_1(G) = 2$.
 Proof: We now claim that $mp_1(G) > 1$. Let H_i be the Hamiltonian cycle of Trampoline Graph. Clearly the cycle H_i is of even length. Let M_1^i consist of alternate edges in H_i and let M_2^i consist of remaining edges in H_i , such that $M_1 = \cup_{i=1}^n M_1^i$ and $M_2 = \cup_{i=1}^n M_2^i$, $1 \leq i \leq n$. Clearly M_1 and M_2 are perfect matching's. Let $e \in H_i$ then either $e \in M_1^i$ or $e \in M_2^i$. Without loss generality $e \in M_1^i$ then M_2^i is a perfect matching. Let E' be the set of edges in $E(G \setminus (M_1 \cup M_2))$. Suppose $e \in E'$, then M_1 and M_2 not containing e . This implies $mp_1(G) > 1$. Hence $mp_1(G) \geq 2$.

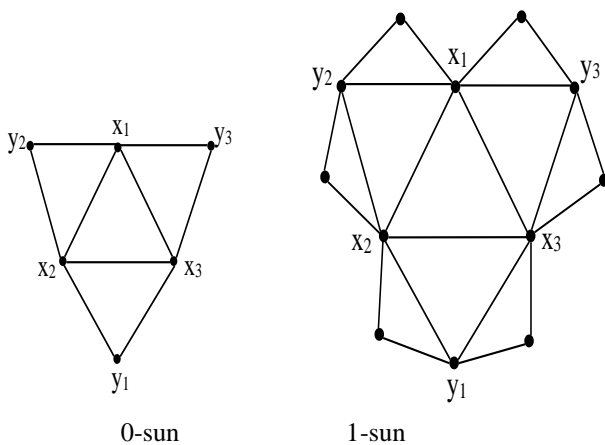


Figure 3: Trampoline Graph

We have only to prove that $mp_1(G) \leq 2$. Now $u-v-w$ is a path in Trampoline Graph with $d_G(u) = 2$, $d_G(v) = 3$ and $d_G(w) = 2$. Remove the edges one end incident to u but not at v , similarly remove the edges one end incident to w but not at v . The resulting graph has uv and vw pendant edges. If the pendant uv is chosen in a perfect matching, the w vertex is left unsaturated and vice versa. Therefore $mp_1(G) = 2$.

VI. DIAMOND SNAKE GRAPH

Definition: A diamond snake graph is obtained by joining vertices v_i and v_{i+1} to two new vertices u_i and w_i for $i = 1, 2, \dots, n - 1$. (see Figure 4) [20].

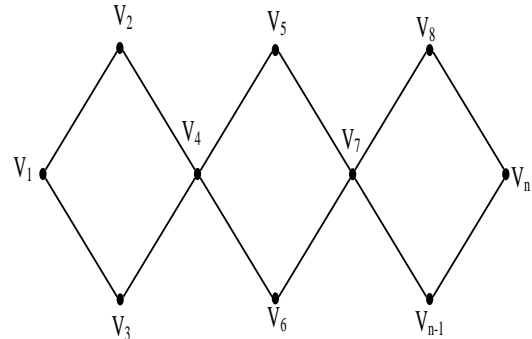


Figure 4: diamond Snake Graph.

Theorem 4: Let G be the Diamond Snake Graph. Then $mp_1(G) = 2$.

Proof: Let M_A be the set of all acute edges and M_O be the set of all obtuse edges in diamond Snake Graph. Thus each of acute, obtuse contains a perfect matching's M_A and M_O respectively. We now claim that $mp_1(G) > 1$. Suppose e_1 in diamond Snake Graph. Clearly e_1 belong to almost one category of edges. But there exists a perfect matching in $G \setminus \{e_1\}$ in the other category of edges. This implies $mp_1(G) > 1$. Hence $mp_1(G) \geq 2$.

We have only to prove that $mp_1(G) \leq 2$. Now $u-v-w$ is a path in triangular ladder with $d_G(u) = 2$, $d_G(v) = 4$ and $d_G(w) = 2$. Remove the edges one end incident to u but not at v , similarly remove the edges one end incident to w but not at v . The resulting graph has uv and vw pendant edges. If the pendant uv is chosen in a perfect matching, the w vertex is left unsaturated and vice versa. Therefore $mp_1(G) = 2$.

VII. K - POLYGONAL SNAKE GRAPH

Definition: Consider n copies of the path P_k where P_k denotes a path graph with K vertices $K \geq 2$. A graph obtained from the path $v_0v_1 \dots \dots v_n$ by identifying the pendant vertices of the i^{th} copy of the path P_k with v_{i-1} and v_i for $i = 1, 2, \dots, n$ is called as K - polygonal snake of length n . (see Figure 5)[22].

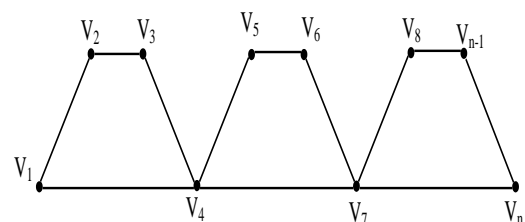


Figure 5: K - Polygonal Snake Graph.

Theorem 5: Let G be the K - Polygonal Snake Graph. Then $mp_1(G) = 2$.

Proof: Let M_H be the set of all horizontal edges and M_{AO} be the set of all acute edges and obtuse edges in K - Polygonal Snake Graph. Thus each of horizontal, acute, obtuse contains a perfect matching's M_{AO} and M_H respectively. We now claim that $mp_1(G) > 1$. Suppose e_1 in K - Polygonal Snake Graph. Clearly e_1 belong to almost one category of edges. But there exists a perfect matching in $G \setminus \{e_1\}$ in the other category of edges. This implies $mp_1(G) > 1$. Hence $mp_1(G) \geq 2$.

We have only to prove that $mp_1(G) \leq 2$. Now $u-v-w$ is a path in K - Polygonal Snake Graph with $d_G(u)=2$, $d_G(v)=4$ and $d_G(w)=2$. Remove the edges one end incident to u but not at v , similarly remove the edges one end incident to w but not at v . The resulting graph has uv and vw pendant edges. If the pendant uv is chosen in a perfect matching, the w vertex is left unsaturated and vice versa. Therefore $mp_1(G) = 2$.

VIII. CONCLUSION

In this paper, we determine the conditional matching preclusion number for Triangular Ladder, C_n with Parallel Chords, Trampoline Graph, Diamond Snake Graph and K -Polygonal Snake Graph. Further determining the conditional matching preclusion for other graphs are under investigation.

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