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Extended Roman Domination Number of Hexagonal Networks

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Abstract - An extended Roman domination function on a graph G=(V,E) is a function $f: V \rightarrow \{0,1,2,3\}$ satisfying the conditions that (i) every vertex u for which f(u) is either 0 or 1 is adjacent to at least one vertex v for which f(v) =3. (ii) if u and v are two adjacent vertices and if f(u)=0 then f(v)≠0, similarly if f(u)=1 then f(v)≠1. The weight of an extended Roman domination function is the value $f(V) = \sum_{u \in V} f(u)$.The minimum weight of an extended Roman domination function on graph G is called the extended Roman domination number of G, denoted by $\gamma_{R_\epsilon}(G)$.The Hexagonal networks are popular mesh-derived parallel architectures. In this paper we present an upper bound for the extended Roman domination number of hexagonal networks.

Keywords: Extended Roman domination, Extended Roman domination number, Hexagonal network.

I. INTRODUCTION

Let *G = (V; E)* be a graph of order *n*. For any vertex $v \in V$, the open neighbourhood of *v* is the set $N(v) = \{u \in V \mid uv \in E\}$ and the closed neighbourhood is the set $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, the open neighbourhood is $N(S) = \bigcup_{v \in S} N(v)$and the closed neighbourhood is $N[S] = N(S) \cup S$. Let $v \in S \subseteq V$. Vertex u is called a private neighbour of *v* with respect to *S* (denotedby u is an *S-pn* of *v*) if $u \in N[v] - N[S - \{v\}]$. An *S-pn* of *v* is external if it is avertex of $V - S$. The set $pn(v; S) = N[v] - N[S - \{v\}]$ of all *S-pn's*of *v* is called theprivate neighbourhood set of *v* with respect to *S*. The set *S* is said to be irredundantif for every $v \in S, pn(v; S) \neq \emptyset$.[2]. A set of vertices S in G is a dominating set, if $N[S] = V(G)$. The domination number, γ (G), of G is the minimum cardinality of a dominating set of G. If S is a subset of V(G), then we denote by G[S] the subgraph of G induced by S. For notation and graph theory terminology in general we follow [4].

In this paper, we study a variant of the domination number called Extended Roman domination number for hexagonal networks.An extended Roman domination function on a graph *G=(V,E)* is a function $f: V \rightarrow \{0,1,2,3\}$satisfying the conditions that (i) every vertex *u* for which *f(u)* is either *0* or *1* is adjacent to at least one vertex *v* for which *f(v) =3*. (ii) if *u* and *v* are two adjacent vertices and if *f(u)=0* then *f(v)≠0*, similarly if *f(u)=1* then *f(v)≠1*.

Cockayne et al. (2004) defined a Roman dominating function (RDF) on $G = (V; E)$to be a function $f : V \rightarrow \{0; 1; 2\}$ satisfying the condition that every vertex u for which $f(u) = 0$ isadjacent to at least one vertex v for which $f(v) = 2$.The definition of a Roman dominating function was motivated by an article in ScientificAmerican by Ian Stewart entitled "Defend the Roman Empire" [5] and suggested even earlier by ReVelle (1997). Each vertex in our graph represents alocation in the Roman Empire. A location (vertex v) is considered unsecured if no legionsare stationed there (i.e., $f(v) = 0$) and secured otherwise (i.e., if $f(v) \in \{1, 2\}$).An unsecured location (vertex v) can be secured by sending a legion to v from an adjacentlocation (an adjacent vertex u). But Constantine the Great (Emperor of Rome)issued a decree in the 4th century A.D. for the defense of his cities. He decreed thata legion cannot be sent from a secured location to an unsecured location if doingso leaves that location unsecured. Thus, two legions must be stationed at a location($f(v) = 2$) before one of the legions can be sent to an adjacent location. In this way,Emperor Constantine the Great can defend the Roman Empire. Since it is expensiveto maintain a legion at a location, the Emperor would like to station as few legions aspossible, while still defending the Roman Empire. A Roman dominating function ofweight $\gamma(G)$ corresponds to such an optimal assignment of legions to locations.

The recent book Fundamentals of Domination in Graphs [10] lists, in an appendix, many varieties of dominating sets that have been studied. It appears that none of those listed are the same as Roman dominating sets. Thus, Roman domination appears to be a new variety of both historical and mathematical interest.

II. PROPERTIES OF EXTENDED ROMAN DOMINATION SETS

For a graph *G=(V,E),* let $f: V \rightarrow \{0,1,2,3\}$ and let (V_0, V_1, V_2, V_3)be the ordered partition of *V* induced by *f*, where $V_i = \{v \in V | f(v) = i\}$and $|V_i| = n_i$, for *i=0,1,2,3*. Note that there exists a 1-1 correspondence between the functions $f: V \rightarrow \{0,1,2,3\}$ and the ordered partitions $(V_0, V_1, V_2, V_3)$of V. Thus, we will write$f = (V_0, V_1, V_2, V_3)$. A function $f = (V_0, V_1, V_2, V_3)$ is an extended Roman domination function if

(i) $V_3 \succ V_0 \cup V_1$,where $\succ$ means that the set V_3 dominates the set $V_0 \cup V_1$, i.e. $V_0 \cup V_1 \subseteq N[V_3]$and (ii) $G(V_0) = \overline{K_{n_0}}$ and $G(V_1) = \overline{K_{n_1}}$, where $G(V_0), G(V_1)$ are the subgraphs induced by V_0 and V_1 respectively. The weight of f is $f(V) = \sum_{u \in V} f(u) = 3n_3 + 2n_2 + n_1$.

We say a function $f = (V_0, V_1, V_2, V_3)$ is a γ_{R_ϵ}- *function* if it is an extended Roman domination function and$f(V) = \gamma_{R_\epsilon}(G)$.[1]

Proposition 1.[1]

For any graph G of order n, $\gamma_{R_\epsilon}(G) = 2\gamma(G)$ if and only if G=$\overline{K}_n$
Proof: It is obvious that if G = $\overline{K_n}$ then $\gamma_{R_\epsilon}(G) = 2\gamma(G)$
Now, assume $\gamma_{R_\epsilon}(G) = 2\gamma(G)$.
Let $f = (V_0, V_1, V_2, V_3)$ be a γ_{R_ϵ}- *function,*

we know, $2\gamma(G) \le 2|V_2| + 2|V_1| \le 3|V_3| + 2|V_2| + |V_1| = \gamma_{R_\epsilon}(G)$.

The equality $\gamma_{R_\epsilon}(G) = 2\gamma(G)$ implies that we have equality in $2\gamma(G) \le 2|V_2| + 2|V_1| = 3|V_3| + 2|V_2| + |V_1| = \gamma_{R_\epsilon}(G)$.
Hence $|V_1| = 0$and$|V_3| = 0$.$|V_3| = 0$implies$V_0 = \emptyset$.Therefore, $\gamma_{R_\epsilon}(G) = 2|V_2| = 2|V| = 2n$.This implies that $2\gamma(G) = 2n \Longrightarrow \gamma(G) = n$, which, inturn, implies that G = $\overline{K_n}$. □

Proposition 2.[1]
Let $f = (V_0, V_1, V_2, V_3)$ be any γ_{R_ϵ}- *function* .Then
a) G (V_2) the subgraph induced by V_2 has max degree 1.
b) $V_2 \cup V_3$is the dominating set for the graph G.
c) V_3 dominates$V_0 \cup V_1$.
d) The subgraph induced by $V_0 \cup V_3$ is either a tree or it is a disconnected graph whose each component is a tree.
e) The subgraph induced by $V_1 \cup V_3$ is either a tree or it is a disconnected graph whose each component is a tree.
f) V_3 is the dominating set for G($V_0 \cup V_1 \cup V_3$).
g) Let H=G($V_0 \cup V_1 \cup V_3$) then each vertex $v \in V_3$ has atleast two H-pn's.(for n>2)

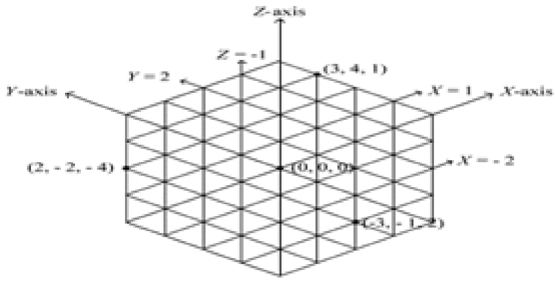
Proposition 3.[1]
For the classes of path P_n,

$$\gamma_{R_E}(P_n) = \begin{cases} \left\lfloor \dfrac{4(n-1)}{3} \right\rfloor + 1 & if\ n \ge 3. \\ \left\lceil \dfrac{4n}{3} \right\rceil & if\ n < 3. \end{cases}$$

Fig.2. Coordinates of vertices in HX(5).

III. UPPER BOUND FOR EXTENDED ROMAN DOMINATION NUMBER OF HEXAGONAL NETWORKS

Hexagonal networks*HX(n)* are multiprocessor interconnection network based on regular triangular tessellations and this is widely studied in [8].Hexagonal networks have been studied in a variety of contexts. They have been applied in chemistry to model benzenoid hydrocarbons [9], in image processing, in computer graphics [7], and in cellular networks [2]. An addressing scheme for hexagonal networks, and its corresponding routing and broadcasting algorithms were proposed by Chen et al.[8].



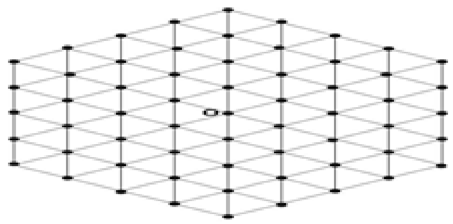


Fig.1. HX(5).

Hexagonal networks *HX(n)* has $3n^2 - 3n + 1$ vertices and $9n^2 - 15n + 6$ edges where n is the number of vertices on one side of the hexagon[8]. The diameter $2n - 2$.There are six vertices of degree three which we call as *corner vertices*. There is exactly one vertex *v* at distance $n - 1$ from each of the corner vertices. This vertex is called the *centre* of *HX(n)* and is represented by *O*. Stojmenovic[6] proposed a coordinate system for a honeycomb network. This was adapted by Nocetti et al.[3] to assign coordinates to the vertices in the hexagonal network. In this scheme, three axes,X,YandZparallel to three edge directions and at mutual angle of 120 degrees between any two of them are introduced, as indicated in Fig.2.We call lines parallel to the coordinate axes asX-lines,Y-lines andZ-lines. HereX=h and X=−k are two X-lineson either side of the X-axis. Any vertex of *HX (n)* is assigned coordinates (x,y,z) in the above scheme. SeeFig.2

Proposition 4.
For a hexagonal network HX *(n)*,$\boldsymbol{\gamma_{R_\epsilon}(HX(n))} \le$

$$\begin{cases} \boldsymbol{4n(n-1)\ \ if\ n \equiv 0\ mod\ 3} \\ \boldsymbol{4n(n-1) + 1\ if\ n \equiv 1\ mod\ 3} \\ \boldsymbol{4n(n-1) - 2\ if\ \ n \equiv 2 mod\ 3} \end{cases}$$

Proof: We will construct anextended Roman dominationfunction for any given hexagonal network. Given any hexagonal network of dimension *n*, we can have the following three cases;

Case (i):
If $\boldsymbol{n \equiv 0\ mod\ 3}$

Consider the center *O* of *HX(n)*, assign it with the value 0. Now consider the vertices on the boundary of *HX(2)* which has O as center(i.e.) the vertices of center hexagon of *HX(n)*. We will assign these vertices the values {1,3} alternatively. Then consider the hexagons that are adjacent to the center hexagon, assign their center verticeswith the value 0. Clearly these hexagons have one edge common with the center hexagon with vertex values 1 and 3. Following this assign the values {1,3} alternatively to the vertices of these hexagons. Repeat this process. Finally we will be left out with semi-hexagons or C_4, these can be minimallylabeled. We observe that $\boldsymbol{n(n-1)}$ vertices are assigned the value 3(i.e.)$|\boldsymbol{V_3}| = \boldsymbol{n(n-1)}$,also another set of $\boldsymbol{n(n-1)}$ vertices are assigned the value 1(i.e.)$|\boldsymbol{V_1}| = \boldsymbol{n(n-1)}$ and remaining vertices are assigned the value zero.Also no vertices are assigned the value 2 (i.e.)$|\boldsymbol{V_2}| = \boldsymbol{0}$.

∴ The weight of this function will be equal to $\boldsymbol{3|V_3| + 2|V_2| + |V_1| = 3n(n-1) + 0 + n(n-1) = 4n(n-1)}$.

Hence, $\boldsymbol{\gamma_{R_\epsilon}(HX(n)) \leq 4n(n-1)\ \ if\ n \equiv 0\ mod\ 3}$.

Case(ii):
If $\boldsymbol{n \equiv 1\ mod\ 3}$
Consider the center *O* of *HX(n)*, assign it with the value 1. Now consider the vertices on the boundary of *HX(2)* which has O as center(i.e.) the vertices of center hexagon of *HX(n)*. We will assign these vertices the values {0,3} alternatively. Then consider the hexagons that are adjacent to the center hexagon, assign their center vertices with the value 1. Clearly these hexagons have one edge common with the center hexagon with vertex values 0 and 3. Following this assign the values {0,3} alternatively to the vertices of these hexagons. Repeat this process. Finally we will be left out with semi-hexagons or C_4, these can be minimally labeled.

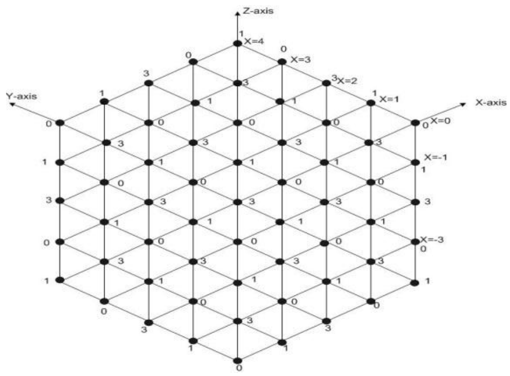


Fig 3.Extended Roman domination function for *HX(5)*.

We observe that $\boldsymbol{n(n-1)}$ vertices are assigned the value 3 (i.e.)$|\boldsymbol{V_3}| = \boldsymbol{n(n-1)}$and $\boldsymbol{n(n-1)+1}$ vertices are assigned the value 1(i.e.)$|\boldsymbol{V_1}| = \boldsymbol{n(n-1)+1}$ and remaining vertices are assigned the value zero. Also no vertices are assigned the value 2 (i.e.)$|\boldsymbol{V_2}| = \boldsymbol{0}$.

∴ The weight of this function will be equal to$\boldsymbol{3|V_3| + 2|V_2| + |V_1| = 3n(n-1) + 0 + (n(n-1)+1) = 4n(n-1) + 1 = 4n(n-1) + 1}$.

Hence,$\boldsymbol{\gamma_{R_\epsilon}(HX(n)) \leq 4n(n-1) + 1\ \ if\ n \equiv 1\ mod\ 3}$.

Case(iii):
If $\boldsymbol{n \equiv 2\ mod\ 3}$

Consider the center *O* of *HX(n)*, assign it with the value 3. Now consider the vertices on the boundary of *HX(2)* which has O as center(i.e.) the vertices of center hexagon of *HX(n)*. We will assign these vertices the values {0,1} alternatively. Then consider the hexagons that are adjacent to the center hexagon, assign their center vertices with the value 3. Clearly these hexagons have one edge common with the center hexagon with vertex values 0 and 3. Following this assign the values {0,1} alternatively to the vertices of these hexagons. Repeat this process. Finally we will be left out with semi-hexagons or C_4, these can be minimallylabeled. We observe that $n(n-1)-1$ vertices are assigned the value 3(i.e.)$|V_3| = n(n-1) - 1$, $n(n-1)+1$ vertices are assigned the value 1(i.e.)$|V_1| = n(n-1)+1$ and remaining vertices are assigned the value zero. Also no vertices are assigned the value 2 (i.e.)$|V_2| = 0$.

∴ The weight of this function will be equal to$3|V_3| + 2|V_2| + |V_1| = 3(n(n-1)-1) + 0 + (n(n-1)+1) = 4n(n-1) - 3 + 1 = 4n(n-1) - 2$.

Hence,$\gamma_{R_\epsilon}(HX(n)) \leq 4n(n-1) - 2\ \ if\ n \equiv 2\ mod\ 3$.□

We observe that this assignment of labeling follows a particular pattern. For any *HX(n)*,the consecutive vertices of $x = 0$ line are assigned the values 1,3,0 respectively. The consecutive vertices of $x = 1$ line are assigned the values 0,1,3 respectively and that of $x = 2$ line are assigned the values 3,0,1 respectively.In general, the consecutive vertices of any $x = i$ lines, where $0 \leq i \leq n$ are assigned the values 3,1,0 if $i \equiv 0\ mod\ 3$, they are assigned the values 0,1,3 if $i \equiv 1\ mod\ 3$ and are assigned the values 3,0,1 if $i \equiv 2\ mod\ 3$.The assignment of values to the vertices of $x = -i$ lines, where $1 \leq i \leq n$is just the reflection of $x = i$ lines. where $1 \leq i \leq n$. See Fig.3.

IV. CONCLUSION

In this paper we have present an upper bound for the extended Roman domination number of hexagonal networks. This work could be further extended to other networks like honeycomb networks, silicate networks, oxide networks, etc.

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