Integrated Intelligent Research (IIR)
International Journal of Computing Algorithm
Vol 02, Issue 01, June 2013, Page No.62-64
ISSN: 2278-2397

Graceful Labeling of Bow Graphs and Shell-Flower Graphs

J. Jeba Jesintha[1], Ezhilarasi Hilda Stanley[2]
[1]PG Department of Mathematics, Women's Christian College, Chennai
[2]Department of Mathematics, Ethiraj College, Chennai.
Email: jjesintha_75@yahoo.com

Abstract - A graceful labeling of a graph G with 'q' edges and vertex set V is an injection f: V(G) → {0,1,2,….q} with the property that the resulting edge labels are also distinct, where an edge incident with vertices u and v is assigned the label |f(u) – f(v)| . A graph which admits a graceful labeling is called a graceful graph. A Shell graph is defined as a cycle C_n with (n -3) chords sharing a common end point called the apex . Shell graphs are denoted as C(n, n- 3). A multiple shell is defined to be a collection of edge disjoint shells that have their apex in common. Hence a double shell consists of two edge disjoint shells with a common apex. A bow graph is defined to be a double shell in which each shell has any order. In this paper we prove that the bow graph with shell orders 'm' and '2m' is graceful. Further in this paper we define a shell – flower graph as k copies of [C(n, n-3) U K_2] and we prove that all shell - flower graphs are graceful for n = 4.

Keywords: Graceful labeling, shell graph , Bow graph,, shell-flower graph.

I. INTRODUCTION

In 1967 Rosa [11] introduced the labeling method called β - valuation as a tool for decomposing the complete graph into isomorphic sub graphs. Later on, this β - valuation was renamed as graceful labeling by Golomb [6]. A *graceful labeling* of a graph G with '*q*' edges and vertex set V is an injection f :V(G) → { 0,1,2,….*q*}with the property that the resulting edge labels are also distinct, where an edge incident with vertices *u* and *v* is assigned the label |f(*u*) – f(*v*)|. A graph which admits a graceful labeling is called a *graceful graph*. Various kinds of graphs are shown to be graceful. In particular, cycle - related graphs have been a major focus of attention for nearly five decades. Rosa[11] showed that the *n - cycle* C_n is graceful if and only if n ≡ 0 or 3 (mod 4). Frucht [4] has shown that the *Wheels* $W_n = C_n + K_1$ are graceful. *Helms* H_n(graph obtained from a wheel by attaching a pendant edge at each vertex of the n – cycle) are shown to be graceful by Ayel and Favaron[1] . Koh, Rogers, Teo and Yap[9] defined a *web graph* as one obtained by joining the pendant points of a helm to form a cycle and then adding a single pendant edge to each vertex of this outer cycle. The *web graph* is proved to be graceful by Kang, Liang, Gao and Yang [8].

Delorme, Koh*et al* [3] showed that any *cycle with achord* is graceful. In 1985 Koh, Rogers, Teo and Yap[10] defined a *cycle with a P_k –chord* to be a cycle with the path P_k joining two non-consecutive vertices of the cycle and proved that these graphs are graceful when k = 3. For an exhaustive survey, refer to the dynamic survey by Gallian[5].

Deb and Limaye [2] have defined a *shellgraph* as a cycle C_n with (n-3) chords sharing a common end point called the *apex*. Shell graphs are denoted as C (n, n-3) (see Figure 1).

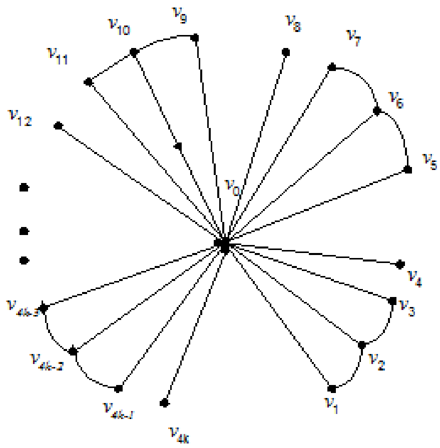

Figure 1. Shell graph C(n, n-3)

Note that the shell C (n, n-3) is the same as the fan $F_{n-1} = P_{n-1} + K_1$. A *multiple shell* is defined to be a collection of edge disjoint shells that have their apex in common. Hence a d*ouble shell* consists of two disjoint shells with a common apex. In [7] a *bow graph* is defined to be a double shell in which each shell has *any* order.

In this paper we prove that all bow graphs with shells of order *m* and *2m* excluding the apex are graceful. Further we define a *shell-flower graph* as 'k 'copies of the union of the shell C (n, n-3) and K_2 where one end vertex of K_2 is joined to the apex of the shell. We denote this graph by $[C(n, n\text{-}3) \cup K_2]^k$ where the superscript k denotes the k copies of [C(n, n-3) U K_2](see Figure 2) and we prove that all shell-flower graphs are graceful when n=4.

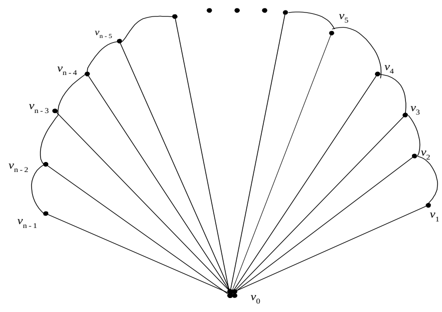

Figure 2. Shell-flower graph $[C(n, n\text{-}3) \cup K_2]^k$ when n = 4

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II. MAIN RESULT

In this section we first prove that bow graphs with shells of order *m* and *(2m)* excluding the apex are graceful and secondly we prove all shell-flower graphs are graceful.

Theorem 1: All Bow graphs with shell orders 'm' and '2m '(order excludes the apex) are graceful.

Proof: Let G be a bow graph with shells of order *m* and *(2m)* excluding the apex. Let the number of vertices in G be '*n*' and the number of edges be '*q*'. We describe the graph G as follows: In G, the shell that is present to the left of the apex is called as the left wing and the shell that is present to the right of the apex is considered as the right wing. Let *m* be the order of the right wing of G and *(2m)*be the order of the left wing of G. The apex of the bow graph is denoted as v_0. Denote the vertices in the right wing of the bow graph from bottom to top as $v_1, v_2, ..., v_m$. The vertices in the left wing of the bow are denoted from top to bottom as $v_{m+1}, v_{m+2}, ..., v_{3m}$. Note that n = (3m +1) and q = (6m - 2). (See Figure. 3)

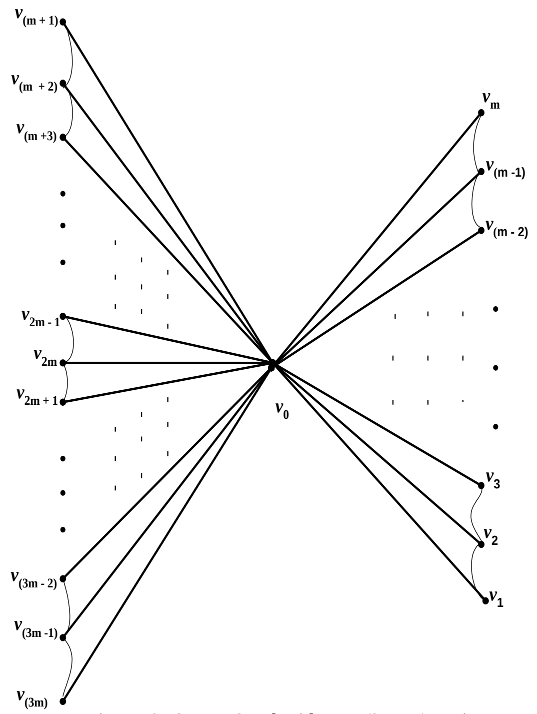

Figure . 3. A Bow Graph with n = (3m + 1) vertices

We label the vertices of the bow graph as follows.

Case 1: When m is odd.

Define

$f(v_0) = 0,$ (1)

$f(v_{2i-1}) = 6m - 2i,$ for $1 \le i \le (m+1)/2$

$5m - 2i + 1,$ for $(m+3)/2 \le i \le (3m+1)/2$ (2)

$f(v_{2i}) = 4m + 2i - 2,$ for $1 \le i \le (m-1)/2$

$3m + 2i - 2,$ for $(m+1)/2 \le i \le (3m-1)/2$ (3)

From the above definition given in (1), (2), (3) we see that the vertices have distinct labels.

We compute the edge labels as follows.

$|f(v_o) - f(v_{2i-1})| = |6m - 2i|,$ for $1 \le i \le (m+1)/2$

$|5m - 2i + 1|,$ for $(m+3)/2 \le i \le (3m+1)/2$ (4)

$|f(v_o) - f(v_{2i})| = |4m + 2i - 2|,$ for $1 \le i \le (m-1)/2$

$|3m + 2i - 2|,$ for $(m+1)/2 \le i \le (3m-1)/2$ (5)

$|f(v_{2i-1}) - f(v_{2i})| = |2m - 4i + 2|,$ for $1 \le i \le (m-1)/2$

$|2m - 4i + 3|,$ for $(m+3)/2 \le i \le (3m-1)/2$ (6)

$|f(v_{2i}) - f(v_{2i+1})| = |2m - 4i|,$ for $1 \le i \le (m-1)/2$

$|2m - 4i + 1|,$ for $(m+1)/2 \le i \le (3m-1)/2$ (7)

From the computations given in (4), (5), (6), (7) we can see that the edge labels are distinct.

Case 2: When m is even.

Define

$f(v_0) = 0$ (8)

$f(v_{2i-1}) = 6m - 2i,$ for $1 \le i \le (m/2)$

$3m + 2i - 3,$ for $(m/2) + 1 \le i \le (3m/2)$ (9)

$f(v_{2i}) = 4m + 2i - 2,$ for $1 \le i \le (m/2)$

$5m - 2i,$ for $(m/2) + 1 \le i \le (3m/2)$ (10)

From the above definition given in (8), (9), (10) we see that the vertices have distinct labels.

We compute the edge labels as follows.

$|f(v_o) - f(v_{2i-1})| = |6m - 2i|,$ for $1 \le i \le (m/2)$

$|3m + 2i - 3|,$ for $(m/2) + 1 \le i \le (3m/2)$ (11)

$|f(v_o) - f(v_{2i})| = |4m + 2i - 2|,$ for $1 \le i \le (m/2)$

$|5m - 2i|,$ for $(m/2) \le i \le (3m/2)$ (12)

$|f(v_{2i-1}) - f(v_{2i})| = |2m - 4i + 2|,$ for $1 \le i \le (m/2)$

$|2m + 4i + 3|,$ for $(m/2) + 1 \le i \le (3m/2)$ (13)

$|f(v_{2i}) - f(v_{2i+1})| = |2m - 4i|,$ for $1 \le i \le (m-2)/2$

$|2m - 4i + 1|,$ for $(m/2) + 1 \le i \le (3m-2)/2$ (14)

From the computations given in (11), (12), (13), (14) one can easily check that the edge labels are distinct. Hence all bow graphs with shell orders 'm' and '2m' are graceful.

The illustrations for both the cases in theorem 1 are shown in the appendix. Next we prove that all s*hell- flower graphs* are graceful. We recall the definition of the shell- flower graph as k copies of the union of the shell C (n, n-3) and K_2 where one end vertex of K_2 is joined to the apex of the shell.

Let G denote the shell-flower graph. Each shell present in G is called as a petal. Hence G has k petals and k pendant edges. Note that G has (4k +1) vertices and 6k edges. We denote the apex as v_0. G comprises of k copies of [C (n, n-3) U K_2]. In the first copy the vertices in the petal are denoted by v1, v_2, v_3 and the pendant vertex is denoted as v_4 . In the 2nd copy the vertices in the petal are denoted as v_5, v_6, v_7 and the pendant vertex is denoted as v_8. The vertices in the other copies are denoted in a similar manner. In general the vertices in the kth copy are (v_{4j+1}, v_{4j+2}, v_{4j+3}), and the pendant vertex is (v_{4j+4}) where j = 0, 1, 2, 3. . . (k-1).

Theorem 2: The shell-flower graphs $[C (n, n-3) U K_2]^k$ are graceful when n = 4.

Proof: Let G be a shell flower graph as described above. We label the vertices of the graph G as follows.

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Define

$$f(v_0) = 0 \qquad (15)$$

$$f(v_{2i-1}) = \begin{cases} 3i - 1, & \text{for } i = 1, 3, 5 \ldots, (2k-1) \\ 3i - 2, & \text{for } i = 2,4,6, \ldots (2k) \end{cases} \qquad (16)$$

$$f(v_{2i}) = \begin{cases} 4m + i, & \text{for } i = 1,,3,5, \ldots (2k-1) \\ 3i, & \text{for } i = 2,4,6, \ldots (2k). \end{cases} \qquad (17)$$

From the above definition given in (15), (16) and (17) we see that the vertices have distinct labels.

We compute the edge labels as follows.

$$|f(v_o) - f(v_{2i-1})| = \begin{cases} 3i - 1, & \text{for } i = 1,3,5, \ldots (2k-1) \\ 3i - 2, & \text{for } i = 2, 4, 6 \ldots (2k). \end{cases} \qquad (18)$$

$$|f(v_o) - f(v_{2i})| = \begin{cases} 4k + i, & \text{for } i = 1,,3,5, \ldots (2k-1) \\ 3i, & \text{for } i = 2, 4, 6, \ldots (2k). \end{cases} \qquad (19)$$

$$|f(v_{2i-1}) - f(v_{2i})| = |4k - 2i + 1| \text{ for } i = 1, 3,5,7, \ldots (2k-1) \qquad (20)$$

$$|f(v_{2i}) - f(v_{2i+1})| = |4k - 2i - 1| \text{ for } i = 1, 3,5,7, \ldots (2k-1) \qquad (21)$$

From the computations given in (18), (19), (20), (21) we can see that the edge labels are distinct. Hence all shell flower graphs $[C(n, n-3) \cup K_2]^k$ are graceful when n=4.The illustration for the shell-flower is given in the appendix.

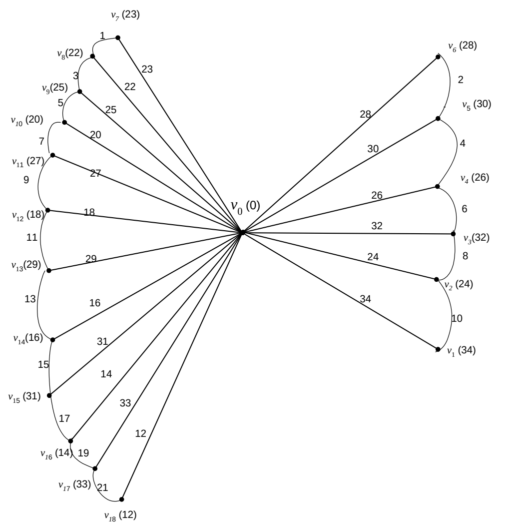

Figure A . Graceful bow graph when m = 6

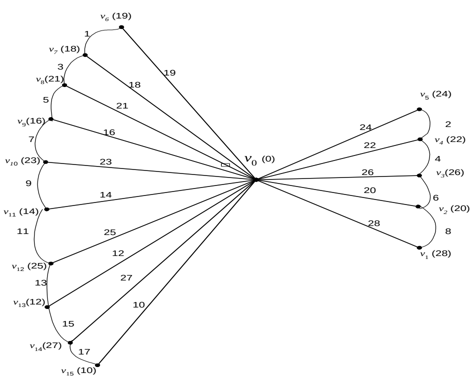

Figure B. Graceful bow graph when m =5

III. CONCLUSION

In this paper we have proved the gracefulness of the bow graph with shell orders 'm' and '2m' and the shell-flower graphs $[C(n, n-3) \cup K_2]^k$ when n = 4.

Appendix

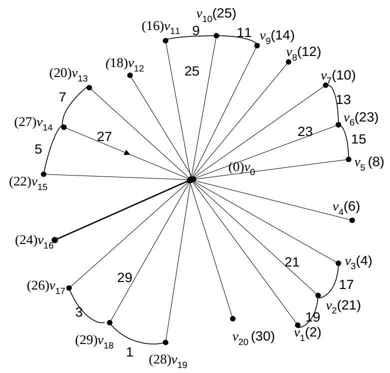

Figure C. Graceful shell- flower graph $[C(n, n-3) \cup K_2]^k$ when n=4 and k =5.

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