Abstract — Let $G_1$ and $G_2$ be two simple graphs. Let $(G_1, \mathcal{F}_1)$ and $(G_2, \mathcal{F}_2)$ be two vertex measure spaces. In this paper we introduce a $\sigma$ algebra $\mathcal{F}_1 \times \mathcal{F}_2$, which consists of all vertex induced sub graphs of $G_1 \times G_2$, and it contains every vertex measurable rectangle graph of the form $H \times K$, $H \in \mathcal{F}_1$ and $K \in \mathcal{F}_2$. Here, we prove $\mathcal{F}_1 \times \mathcal{F}_2$ is the smallest $\sigma$ algebra of $G_1 \times G_2$ such that the maps $P_x: X \times Y \to X$ and $P_y: X \times Y \to Y$ defined by $P_x(x, y) = x$ and $P_y(x, y) = y$ respectively are measurable. In this paper we develop the graph analog of these concepts.

Keywords — vertex measurable graph, vertex measurable rectangle graph

I. INTRODUCTION

The authors [3] introduced the concept of vertex measurable graph and proved some results related to this concept. The authors [4] introduced a new operation ‘$\times$’ as in definition 2.4 and vertex measurable rectangle graph. The concept of Cartesian product of two measurable spaces was introduced in the field of measure theory. In [2] it has been proved that $\mathcal{F}_1 \times \mathcal{F}_2$ is the smallest $\sigma$-algebra of subsets of $X \times Y$ such that the maps $P_x: X \times Y \to X$ and $P_y: X \times Y \to Y$ defined by $P_x(x, y) = x$ and $P_y(x, y) = y$ respectively are measurable. In this paper we develop the graph analog of these concepts.

II. PRELIMINARIES

Definition 2.1
A graph $G$ with $p$ vertices and $q$ edges is called a $(p, q)$ graph, where $p$ and $q$ are respectively known as the order and size of the graph $G$.

A $(p, q)$ graph with $p = q = 0$ is called an empty graph and is denoted by $\Phi$.

Definition 2.2
Let $G = (V_G, E_G)$ be a graph and $H = (V_H, E_H)$ be a sub graph of $G$. The vertex complement of $H$ in $G$ is denoted by $H^c$ and it is defined as the sub graph obtained from $G$ by deleting all the vertices of $H$. Hereafter, we shall use $H^c$ instead of $H^c_G$.

From the examples below it is evident that $H \cup H^c$ is not equal to $G$. So there is a hidden sub graph of $G$ related to $H$ to overcome this difficulty, a new union was defined in [3].

Definition 2.3
Let $G = (V_G, E_G)$ be a graph and for $S_1 \subseteq V_G$ and $S_2 \subseteq V_G$. Let $H_1 = (S_1)$ and $H_2 = (S_2)$ be two vertex induced sub graphs of $G$. The vertex induced union of $H_1$ and $H_2$ is defined as the vertex induced sub graph $(S_1 \cup S_2)$ and is denoted by $H_1 \cup H_2$.

Definition 2.4
If $H_1 = (S_1)$ and $H_2 = (S_2)$ are two sub graphs of $G$ then $H_1 \cup H_2 = (S_1 - S_2)$.

Definition 2.5
Let $G$ be a simple graph and let $\mathcal{F}$ be a collection of vertex induced sub graphs of $G$ together with empty graph $\Phi$ is a field if and only if

(i) $G \in \mathcal{F}$
(ii) $H^c \in \mathcal{F}$ for each $H \in \mathcal{F}$
(iii) $H_1, H_2, \ldots \in \mathcal{F}$ then $H_1 \cup H_2 \in \mathcal{F}$ and $\mathcal{F}$

Example 2.7
Let $G_1 = (V_{G_1}, E_{G_1})$ and $G_2 = (V_{G_2}, E_{G_2})$ be two simple graphs. The Cartesian product of $G_1$ and $G_2$ denoted as $G_1 \times G_2$ is a graph with vertex set $V_{G_1} \times V_{G_2}$ and two vertices $u = (u_1, v_1)$ and $v = (u_2, v_2)$ are said to be adjacent if $u_1 = u_2$ and $v_1$ is adjacent to $v_2$ in $G_2$ or $v_1 = v_2$ and $u_1$ is adjacent to $u_2$ in $G_1$. That is, $G_1 \times G_2 = (V_{G_1} \times V_{G_2}, E_{G_1} \times E_{G_2})$ where $E_{G_1} \times E_{G_2} = \{ uv / u_1 = u_2 \text{ and } v_1v_2 \in E_{G_2} \text{ or } v_1 = v_2 \text{ and } u_1u_2 \in E_{G_1} \}$. 

The Cartesian product of vertex measurable graphs was introduced in [6]. The authors [7] introduced the concept of vertex measurable graph, vertex measurable rectangle graph.
III. VERTEX MEASURABLE RECTANGLE GRAPH

Definition 3.1
Let $G_1$ and $G_2$ be any two simple graphs. Let $(G_1, \mathcal{A}_1)$ and $(G_2, \mathcal{A}_2)$ be two vertex measurable spaces. Any graph of the form $H_1 \times H_2$, where $H_1 \in \mathcal{A}_1$ and $H_2 \in \mathcal{A}_2$, is called a vertex measurable rectangle graph.

Definition 3.2
Let $G_1$ and $G_2$ be any two simple graphs. Let $N$ be any vertex induced sub graph of $G_1 \times G_2$. For each vertex $u$ in $G_1$, define $N^u = \{v_i / v_i \in V(G_1) \text{ and } (u, v_i) \in V(N)\}$, where $V(G_2)$ is the vertex induced sub graph from the vertex set $V(G_1)$. Clearly $N^u$ is a sub graph of $G_2$. Similarly it is denoted as $N^v$.

Similarly for each vertex $v$ in $G_2$ define $V(G_2)\times V(G_1)$ where $V(G_1) = \{v_i / v_i \in V(G_1) \text{ and } (u, v_i) \in V(N)\}$, $(V(G_2)\times V(G_1))$ is the vertex induced sub graph from the vertex set $V(G_2)\times V(G_1)$. Clearly $N^u$ is a sub graph of $G_2$. Simply it is denoted as $N^v$.

Lemma 3.3
Let $(G_1, \mathcal{A}_1)$ and $(G_2, \mathcal{A}_2)$ be two vertex measure spaces. If $N = H \times K$ is a vertex measurable rectangle graph in $\mathcal{A}_1 \times \mathcal{A}_2$, then $N^u = \{V(K) \times \mathcal{A}_2\}$ where $V(N) = \{v_i / v_i \in V(G_1) \text{ and } (u, v_i) \in V(N)\}$, $(u, v_i) \in V(N)$.

Proof:
Let $u \in V(H)$.
Then $N^u = \{V(K) \times \mathcal{A}_2\}$ where $V(N) = \{v_i / v_i \in V(G_1) \text{ and } (u, v_i) \in V(N)\}$, $(u, v_i) \in V(N)$.

Hence $N^u = K$ if $u \in V(H)$.
If $u$ is not in $V(H)$, then $N^u = \{V(K)\}$ where $V(N) = \{v_i / v_i \in V(G_1)\}$ and $(u, v_i) \in V(N)$.

Therefore, $N^u = \{K\}$ if $u \in V(H)$.
Hence, $N^u = \{\phi\}$ if $u \notin V(H)$.

Similarly we can prove for $v \in V(K)$.

Lemma 3.4
If $H_1 \times K_1$ and $H_2 \times K_2$ are two vertex measurable rectangle graphs in $\mathcal{A}_1 \times \mathcal{A}_2$, then $(H_1 \times K_1) \cap (H_2 \times K_2) = (H_1 \cap H_2) \times (K_1 \cap K_2)$.

Proof:
For the proof, we claim that $V((H_1 \times K_1) \cap (H_2 \times K_2)) = V((H_1 \cap H_2) \times (K_1 \cap K_2))$ and $E((H_1 \times K_1) \cap (H_2 \times K_2)) = E((H_1 \cap H_2) \times (K_1 \cap K_2))$.

Now, $V((H_1 \times K_1) \cap (H_2 \times K_2)) = V((H_1 \times K_1) \cap (H_2 \times K_2)) = V((H_1 \cap H_2) \times (K_1 \cap K_2))$.

Therefore, $V((H_1 \times K_1) \cap (H_2 \times K_2)) = V((H_1 \cap H_2) \times (K_1 \cap K_2))$.

Let $e = uv \in E((H_1 \times K_1) \cap (H_2 \times K_2))$ where $u = (u_1, v_1)$ and $v = (u_2, v_2)$. Then $u_1 \in V(H_1)$ and $v_1 \in V(K_1)$.

By the definition of Cartesian product,

$u_1 = u_2 \text{ and } v_1 = v_2 \text{ or } u_1 = u_2 \text{ and } v_1 \neq v_2 \text{ and } u_1 \neq u_2$.

that implies and implied by

$u_1 = u_2 \text{ and } v_1 = v_2 \text{ or } u_1 = u_2 \text{ and } v_1 \neq v_2 \text{ and } u_1 \neq u_2$.

or

$u_1 = u_2 \text{ and } v_1 \neq v_2 \text{ and } u_1 \neq u_2$.

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or

$u_1 = u_2 \text{ and } v_1 \neq v_2 \text{ and } u_1 \neq u_2$.

that implies and implied by

$u_1 = u_2 \text{ and } v_1 = v_2 \text{ or } u_1 = u_2 \text{ and } v_1 \neq v_2 \text{ and } u_1 \neq u_2$.

or

$u_1 = u_2 \text{ and } v_1 \neq v_2 \text{ and } u_1 \neq u_2$.
Let \( G_1 \) and \( G_2 \) be two vertex measure spaces. Let \( L_{G_1}: G_1 \times G_1 \rightarrow G_1 \) and \( L_{G_2}: G_2 \times G_2 \rightarrow G_2 \) be defined by \( L_{G_1}(H \times K) = H \) and \( L_{G_2}(H \times K) = K \) for all vertex measurable graphs \( H \) in \( G_1 \) and \( K \) in \( G_2 \), Then

(i) The maps \( L_{G_1} \) and \( L_{G_2} \) are measurable.

(ii) The \( \sigma \)-algebra \( \mathcal{G}_1 \times \mathcal{G}_2 \) is the smallest \( \sigma \)-algebra of \( G_1 \times G_2 \), such that (i) holds.

**Proof:**

Let \( G_1 \) and \( G_2 \) be two simple graphs. Let \( (G_1, \mathcal{G}_1) \) and \( (G_2, \mathcal{G}_2) \) be two vertex measure spaces. Let \( H \in \mathcal{G}_1 \) and \( K \in \mathcal{G}_2 \). By lemma 3.3, for any vertex measurable graph \( H \times K \) in \( \mathcal{G}_1 \times \mathcal{G}_2 \),

\[
(H \times K)^u = \begin{cases} K & \text{if } u \in V(H) \\ \phi & \text{if } u \in V(H) \end{cases}
\]

for all \( u \in V(H) \). This follows that \( L_{G_1}^{-1}(H) \in \mathcal{G}_1 \times \mathcal{G}_2 \) and \( L_{G_2}^{-1}(K) \in \mathcal{G}_1 \times \mathcal{G}_2 \).

Let \( \Omega \) be any \( \sigma \)-algebra of vertex induced sub graphs of \( G_1 \times G_2 \) such that \( L_{G_1} \) and \( L_{G_2} \) are vertex measurable. We have to show that \( \Omega \subseteq \mathcal{G}_1 \times \mathcal{G}_2 \). Let \( H \in \mathcal{G}_1 \) and \( K \in \mathcal{G}_2 \). By the definition of \( L_{G_1} \) and \( L_{G_2} \), \( L_{G_1}(H \times G_2) = H \) which is \( H \times G_2 = L_{G_2}^{-1}(H) \). Since, \( H \times G_2 \in \Omega \), \( L_{G_2}^{-1}(H) \in \Omega \). Similarly \( L_{G_2}^{-1}(K) \in \Omega \). By lemma 3.4 \( H \times K = (H \cap G_2) \cup (K \cap G_2) = (H \times G_2) \cap (G_2 \times K) \). It follows that \( \mathcal{G}_1 \times \mathcal{G}_2 \subseteq \Omega \). Hence \( \mathcal{G}_1 \times \mathcal{G}_2 \) is the smallest \( \sigma \)-algebra of vertex measurable graphs of \( G_1 \times G_2 \).

**References**


