Stability of Duodecic Functional Equations in Multi-Banach Spaces

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Abstract - In this paper, we carry out the Hyers- Ulam Stability of duodecic functional equation in Multi-Banach Spaces by using fixed point method.

I. INTRODUCTION

In 1940, Ulam posed a problem concerning the stability of functional equations: Give conditions in order for a linear function near an approximately linear function to exist. An earlier work was done by Hyers [6] in order to answer Ulam’s equation [13] on approximately additive mappings. During last decades various stability problems for large variety of functional equations have been investigated by several mathematicians. A large list of references concerning in the stability of functional equations can be found. e.g.[11, 6, 7, 8, 9, 14].

In this paper, we carry out the Hyers-Ulam stability of duodecic functional equation

$$G(x; y) = f(x + 6y) − 12f(x + 5y) + 66f(x + 4y) − 220f(x + 3y) + 495f(x + 2y) − 792f(x + y) + 924f(x) − 792f(x − y) + 495f(x − 2y) − 220f(x − 3y) + 66f(x − 4y) − 12f(x − 5y) + f(x − 6y) − 12! f(y)$$

where $12! = 479001600$, in Multi-Banach Spaces by using fixed point approach. It is easily verified that that the function $f(x) = x^{12}$ satisfies the above functional equations. In other words, every solution of the duodecic functional equation is called a dodecic mapping. Now, we present the following theorem due to B. margolis and J.B. Diaz [3], [11] for the fixed point theory.

Theorem 1 [3], [11] Let $(x, d)$ be a complete generalized metric space and let $f : X \rightarrow X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each given element $x \in X$, either $f^n(x)$, $f^{n+1}(x) = \infty$

for all nonnegative integers $n$ or there exists a positive integer $n_0$ such that

\begin{align*}
&i. \quad d(f^n(x), f^{n+1}(x)) < \infty \quad \text{for all} \quad n \geq n_0; \\
&ii. \quad (ii) \quad \text{The sequence} \; f^n(x) \; \text{is convergent to a fixed point} \; y^* \; \text{of} \; f; \\
&iii. \quad y^* \; \text{is the unique fixed point of} \; T \; \text{in the set} \; \{y \in X : d(f^n(x), y) < \infty\}; \\
&iv. \quad d(y, y^*) \leq \frac{1}{1-L} d(y, f(y)) \; \text{for all} \; y \in Y.
\end{align*}

Theorem 2

Let $A$ be a linear space and let $((\ell^k, \| \cdot \|_k) : k \in \mathbb{N})$ be a multi-Banach space. Suppose that $\eta$ is a non-negative real number and $f : A \rightarrow B$ is a mapping fulfills

$$sup_{x \in X} \|G(f(x_1), y_1), \ldots, G(f(x_k), y_k)\|_k \leq \eta$$

\begin{equation}
(1)
\end{equation}

For all $x_1, x_2, \ldots, x_k, y_1, \ldots, y_k \in A$. Then there exists a unique duodecic mapping $D : A \rightarrow B$ such that

$$sup_{x \in X} \|f(x_1) − D(x_1), \ldots, f(x_k) − D(x_k)\|_k \leq \eta$$

\begin{equation}
(2)
\end{equation}

For all $x_1, x_2, \ldots, x_k \in A$.\n
Proof.

Doing $x_1 = x_2 = \ldots = x_k = 0$ and changing $y_1, \ldots, y_k$ by $2x_1, \ldots, 2x_k$ in (1), and dividing by 2 in the resulting equation, we get

$$sup_{x \in X} \|f(12x_1) − 12f(10x_1) + 66f(8x_1) − 220f(6x_1) + 495f(4x_1) − 239501592f(2x_1), \ldots, f(12x_k) − 12f(10x_k) + 66f(8x_k) − 220f(6x_k) + 495f(4x_k) − 239501592f(2x_k)\|_k \leq \frac{\eta}{2}$$

\begin{equation}
(3)
\end{equation}

for all $x_1, x_2, x_k \in A$. Taking $x_1, x_k$ by $6x_1, \ldots, 6x_k$ and changing $y_1, y_2, \ldots, y_k$ by $x_1, \ldots, x_k$ in (1), we have

$$sup_{x \in X} \|f(12x_1) − 12f(10x_1) + 66f(8x_1) − 220f(9x_1) + 495f(8x_1) − 792f(7x_1) + 924f(6x_1) − 792f(5x_1) + 495f(4x_1) − 220f(3x_1) + 66f(2x_1) − 792f(5x_k) + 924f(4x_k) − 220f(3x_k) + 66f(2x_k) − 479001612f(x_k)\|_k \leq \eta$$

\begin{equation}
(4)
\end{equation}

for all $x_1, \ldots, x_k \in A$. Combining (3) and (4), one gets

$$sup_{x \in X} \|f(12(x_1) − 12f(10(x_1) + 66f(9(x_1) − 220f(9(x_1) + 495f(8(x_1) − 792f(7(x_1) + 924f(6(x_1) − 792f(5(x_1) + 495f(4(x_1) − 220f(3(x_1) + 66f(2(x_1) − 792f(5(x_k) + 924f(4(x_k) − 220f(3(x_k) + 66f(2(x_k) − 479001612f(x_k)\|_k \leq \frac{\eta}{2}$$

\begin{equation}
(5)
\end{equation}

for all $x_1, \ldots, x_k \in A$. Taking $x_1, \ldots, x_k$ by $5x_1, \ldots, 5x_k$ and changing $y_1, y_2, \ldots, y_k$ by $x_1, \ldots, x_k$ in (1), we have

$$sup_{x \in X} \|f(11x_1) − 12f(10x_1) + 66f(9x_1) − 220f(8x_1) + 495f(7x_1) − 792f(6(x_1) + 924f(5(x_1) − 792f(4(x_1) + 495f(3(x_1) − 220f(2(x_1) + 479001533f(x_1) − 12f(10x_k) + 66f(9(x_k) − 220f(8(x_k) + 495f(7(x_k) − 239501592f(2x_k)\|_k \leq \frac{\eta}{2}$$

\begin{equation}
(6)
\end{equation}

for all $x_1, \ldots, x_k \in A$.
for all $x_1,\ldots, x_k \in A$. Multiplying by 12 on both sides of (6), then it follows from (5) and the resulting equation, we arrive at

$$\sup_{k \in \mathbb{N}} \left( \frac{1}{4096} f(x_k) - f(x_i), \ldots, \frac{1}{4096} f(x_k) - f(x_i) \right) \leq \frac{1}{2} \ell \lambda$$

for all $x_1,\ldots, x_k \in A$.

Hence, it holds that

$$d(f,l,m) = \frac{1}{2}\ell \lambda d(I, f)$$

for all $l, m \in A$. We assert that $J$ is a strictly contractive operator. Given $l, m \in A$, let $\lambda \in [0,\infty)$ be an arbitrary constant with $d(J, l, m) \leq \lambda$. From the definition it follows that

$$\sup_{k \in \mathbb{N}} \left( J(x_k) - J(x_i), \ldots, J(x_k) - J(x_i) \right) \leq \frac{1}{2} \ell \lambda$$

This completes the proof of the Theorem.

References